Adjoint pairs of functors and Frobenius extensions

By

Kiiti MORITA

(Received May 10, 1965)

Introduction

Throughout this paper A and B are assumed to be associative rings which possess identity elements 1_A and 1_B respectively. The category of all left (resp. right) A-modules will be denoted by ${}_A\mathfrak{M}$ (resp. $\mathfrak{M}_A$), and by a functor we shall always mean a covariant additive functor. In case we speak of a subring B of A we shall assume that $1_B=1_A$. All modules over a ring are assumed to be unitary.

Let S be a functor from ${}_A\mathfrak{M}$ to ${}_B\mathfrak{M}$ and T a functor from ${}_B\mathfrak{M}$ to ${}_A\mathfrak{M}$. In case there is a natural isomorphism

$$(1) \qquad \mathrm{Hom}_B(S(X), Y)\cong\mathrm{Hom}_A(X, T(Y)) \qquad \text{for } X\in{}_A\mathfrak{M},\ Y\in{}_B\mathfrak{M},$$

where both sides of (1) are considered as bifunctors in X and Y with values in the category of abelian groups, we say, following D. M. Kan [5], that S is a left adjoint of T and T is a right adjoint of S. If S is a left adjoint of T and T has a right adjoint we shall say that $\{S, T\}$ is an adjoint pair of functors.

Among adjoint pairs of functors the simplest one is such that $\{T, S\}$ as well as $\{S, T\}$ is an adjoint pair; in this case $\{S, T\}$ will be called a strongly adjoint pair. As an example of a strongly adjoint pair we can mention category-isomorphisms; if ST and TS are naturally equivalent to the identity functor, each of S and T is called a category-isomorphism[1].

Another example of strongly adjoint pairs is provided by Frobenius extensions. The notion of Frobenius extensions was first introduced by F. Kasch [6] and later generalized by T. Nakayama and T. Tsuzuku [14], and by Kasch himself [7]. Let B be a subring of A. Then it will be shown (§ 5 below) that A is a Frobenius extension of B in the sense of Kasch [7] if and only if the functors S and T defined by $S(X)={}_BA_A\otimes X$, $T(Y)={}_AA_B\otimes Y$ form a strongly adjoint pair.

The notion of β-Frobenius extensions in the sense of Nakayama-Tsuzuku [14] involves an automorphism β of the subring B of A. This notion, however, exhibits a midway feature, as is pointed out in [14, Part II, p. 137], in generalizing the endomorphism ring theorem of Kasch; indeed, Theorem 22 in [14] necessitates a rather strong assumption that β be extendable to an automorphism of A. On the other

1) In this case T is called an inverse of S and will be denoted by S^{-1}; S^{-1} is determined uniquely by S up to a natural equivalence. As for category-isomorphisms, cf. K. Morita [10], [11], H. Bass [2]; [2] gives a nice summary of the theory.

hand, in [14, Part I, p. 92] Nakayama and Tsuzuku suggested generalizing the notion by taking account of an automorphism α of A besides β. Their intention seems to be realized by the notion of (α, β)-Frobenius extensions defined below, which, however, has still a midway feature[2]. The unsatisfactory point seems to be remedied by considering the situation from the point of view of adjoint pairs of functors.

Let S_0 be a category-isomorphism from ${}_A\mathfrak{M}$ to itself and T_0 a category-isomorphism from ${}_B\mathfrak{M}$ to itself. In case $\{S, T\}$ is an adjoint pair of functors such that $T_0^{-1}SS_0$ is a right adjoint of T, we shall say that $\{S, T\}$ is an (S_0, T_0)-strongly adjoint pair. In § 5 we define the notion of (S_0, T_0)-Frobenius extensions so that A is an (S_0, T_0)-Frobenius extension of B if and only if $\{T_0S'S_0^{-1}, T\}$ is an (S_0, T_0)-strongly adjoint pair where $S'(X)={}_BA_A\otimes X$, $T(Y)={}_AA_B\otimes Y$, for $X\in{}_A\mathfrak{M}$, $Y\in{}_B\mathfrak{M}$. In case S_0 is a category-isomorphism induced by an automorphism α of A and T_0 is one induced by an automorphism β of B, we speak of an (α, β)-Frobenius extension instead of an (S_0, T_0)-Frobenius extension; a β-Frobenius extension in the sense of [14] is nothing but a $(1, \beta)$-Frobenius extension in our sense.

The purpose of the present paper is to develop a theory of adjoint pairs of functors. Our first result, Theorem 3.1 in § 3, asserts that $\{S, T\}$ is an adjoint pair of functors if and only if there are natural equivalences[3]:

$$S(X)\cong{}_BU_A\otimes X,\ T(Y)\cong{}_AV_B\otimes Y, \qquad X\in{}_A\mathfrak{M},\ Y\in{}_B\mathfrak{M},$$

with bimodules ${}_BU_A$, ${}_AV_B$ such that

(2) ${}_BU$ is finitely generated and projective[4],

(3) ${}_AV_B\cong{}_A[\mathrm{Hom}_B({}_BU_A, {}_BB)]_B$[5].

In § 4 we shall define (S_0, T_0)-quasi-strongly adjoint pairs and in § 5 the notion of (S_0, T_0)-quasi-Frobenius extensions will be introduced. Quasi-Frobenius algebras were introduced by T. Nakayama as a generalization of Frobenius algebras. While the notion of Frobenius extensions was obtained about ten years ago, it is quite recently that the notion of quasi-Frobenius extensions has been defined by B. Müller [13] and by A. Rosenberg-S. Chase independently. Quasi-Frobenius extensions in their sense are special cases of (S_0, T_0)-quasi-Frobenius extensions in our sense.

The well-known theorem of Gaschütz-Ikeda-Kasch concerning relatively projective and injective modules will be discussed in the framework of adjoint pairs of functors (§ 6).

A left A-module is called reflexive (cf. [1]) if the natural A-homomorphism

(4) $$\pi(X):\ X\to\mathrm{Hom}_A(\mathrm{Hom}_A(X, {}_AA), A_A)$$

2) Even if A is a $(1, \beta)$-Frobenius extension of B, the B-endomorphism ring of A_B is not always a (γ, α)-Frobenius extension of A except for the case where A_B is free or $\beta=1$.

3) In case both sides of the symbol $\cong$ are considered as functors, by $\cong$ we mean "is naturally equivalent to".

4) As usual, a module M is written as ${}_AM$ (resp. M_A) in case it is to be stressed that M is considered as a left (resp. right) A-module; similarly for ${}_AM_B$.

5) For $a\in A$, $b\in B$, $f\in\mathrm{Hom}_B(U, B)$, we define af and fb, as useal, by $(af)(u)=f(ua)$, $(fb)(u)=f(u)b$, $u\in U$.

is an A-isomorphism. In § 7 we shall prove that in case $\{S, T\}$ is an adjoint pair satisfying certain conditions, X is reflexive if and only if the left B-module $S(X)$ is reflexive. This constitutes a generalization of the duality in quasi-Frobenius algebras. In § 8 it will be shown that Nakayama isomorphism can be defined for adjoint pairs satisfying some conditions.

The endmorphism ring theorem will be established in § 9 for (S_0, T_0)-quasi-strongly adjoint pairs.

Finally, in § 10 we shall give a characterization of category-isomorphisms for the case where A and B satisfy the minimum condition, from the stand point of adjoint pairs of functors.

§ 1. Preliminaries

A left A-module V is called a *generator* (of ${}_A\mathfrak{M}$) if there is a positive integer n such that a direct sum of n copies of V has a direct summand which is A-isomorphic to ${}_AA$; V is a generator if and only if there exist a finite number of elements $v_i \in V$, $\varphi_i \in \mathrm{Hom}_A(V, {}_AA)$, $i=1,\cdots,n$, such that $\sum_{i=1}^{n}\varphi_i(v_i)=1_A$.

For a right A-module U_A we denote by $\mathrm{End}_A(U_A)$ the A-endomorphism ring of U_A; we consider $\mathrm{End}_A(U_A)$ as a left operator domain of U_A. For a left B-module ${}_BU$ we denote by $[\mathrm{End}_B({}_BU)]^0$ the ring which is inverse-isomorphic to the B-endomorphism ring of ${}_BU$; we consider $[\mathrm{End}_B({}_BU)]^0$ as a right operator domain of ${}_BU$. In case ${}_BU_A$ is a B-A-bimodule, by $B=\mathrm{End}_A(U_A)$ we shall mean that B is isomorphic to $\mathrm{End}_A(U_A)$ by the correspondence $b\to\phi_b$ where $\phi_b(u)=bu$ for $u\in U$; similarly by $A=[\mathrm{End}_B({}_BU)]^0$ it is meant that A is inverse-isomorphic to $\mathrm{End}_B({}_BU)$ by the correspondence $a\to\varphi_a$ where $\varphi_a(u)=ua$ for $u\in U$.

LEMMA 1.1. *Let ${}_AV_B$ be an A-B-bimodule. Then the following statements hold.*

1) *If ${}_AV$ is a generator and $B=[\mathrm{End}_A({}_AV)]^0$, then V_B is finitely generated and projective and $A=\mathrm{End}_B(V_B)$.*

2) *If V_B is finitely generated and projective and if $A=\mathrm{End}_B(V_B)$, then ${}_AV$ is a generator.*

This is nothing else Morita [10, Lemma 3.3].

THEOREM 1.2. *If $S: {}_A\mathfrak{M}\to{}_B\mathfrak{M}$ is a category-isomorphism, there exists a B-A-bimodule ${}_BU_A$ such that there is a natural equivalence $S(X)\cong{}_BU_A\otimes X$. For a B-A-bimodule ${}_BU_A$ the following conditions are equivalent.*

I. *The functor ${}_BU_A\otimes X$ from ${}_A\mathfrak{M}$ to ${}_B\mathfrak{M}$ is a category-isomorphism.*

II. *U_A and ${}_BU$ are finitely generated and projective, and $B=\mathrm{End}_A(U_A)$, $A=[\mathrm{End}_B({}_BU)]^0$.*

III. *U_A is a finitely generated, projective generator and $B=\mathrm{End}_A(U_A)$.*

This is proved in Morita [10, Theorems 3.2, 3.4 and Lemma 3.3]. See also H. Bass [2]. Theorem 3.1 in § 3 below is viewed as a generalization of Theorem 1.2.

LEMMA 1.3. *Let ${}_AV_B$ be an A-B-bimodule. If ${}_AV$ is a generator, then ${}_AV_B\otimes Y$ is a generator of ${}_A\mathfrak{M}$ for any generator Y of ${}_B\mathfrak{M}$.*

Proof is obvious.

LEMMA 1.4. *Let B be a subring of A and V a left A-module. If ${}_BV$ is a gen-*

erator, then ${}_BB$ is a direct summand of ${}_BA$.

PROOF[6]. By assumption there exist $v_i \in V$, $\varphi_i \in \mathrm{Hom}_B(V, {}_BB)$, $i=1,\cdots,n$, such that $\sum_{i=1}^{n} \varphi_i(v_i)=1_B$. Let us set

$$\psi(a)=\sum_{i=1}^{n} \varphi_i(av_i), \qquad \text{for } a \in A.$$

Then $\psi \in \mathrm{Hom}_B({}_BA, {}_BB)$ and $\psi(b)=b$ for $b \in B$.

A functor S from ${}_A\mathfrak{M}$ to an abelian category is said to be *faithful* if $S(f)=0$ implies $f=0$ for $f \in \mathrm{Hom}_A(X, X')$ (S. MacLane [9, p. 263]).

LEMMA 1.5. *Let V be a left A-module and set $S'(X)=\mathrm{Hom}_A(V, X)$, $X \in {}_A\mathfrak{M}$. Then the functor S' is faithful if and only if ${}_AV$ is a generator. If S' is faithful then $S'(X)=0$ implies $X=0$; conversely, in case V is projective, if $S'(X)=0$ implies $X=0$ then S' is faithful*[6a].

PROOF. Suppose that $S'(f)=0$ for $f \in \mathrm{Hom}_A(X, X')$. Let $x_0 \in X$. If there are $v_i \in V$, $\varphi_i \in \mathrm{Hom}_A(V, A)$, $i=1,\cdots,n$, such that $\sum \varphi_i(v_i)=1$, then $f(x_0)=\sum (f \circ g_i)(v_i)=0$ where we set $g_i(v)=\varphi_i(v)x_0$ for $v \in V$. Thus $f=0$, as desired.

Let J be the left ideal of A generated by $\varphi(v)$ for all $v \in V$, $\varphi \in \mathrm{Hom}_A(V, A)$. Let σ be the canonical projection of ${}_AA$ onto A/J. Suppose that S' is faithful (resp. V is projective and $S'(X)=0$ implies $X=0$). Then $S'(\sigma)=0$ (resp. $S'(A/J)=0$). Hence $J=A$, that is, V is a generator.

§ 2. Dual sets of generators

For $U \in {}_B\mathfrak{M}$, $V \in \mathfrak{M}_B$, by a *B-bilinear form on $U \times V$* we shall mean a map ω: $U \times V \to B$ such that ω is additive with respect to $u \in U$ and $v \in V$ and $\omega(bu, v)=b\omega(u, v)$, $\omega(u, vb)=\omega(u, v)b$ for $b \in B$, $u \in U$, $v \in V$.

Two sets of elements, $\{u_i \in U, i=1,\cdots,n\}$ and $\{v_i \in V, i=1,\cdots,n\}$, will be called *dual sets of generators* of U and V with respect to a B-bilinear form ω, if the conditions (5) and (6) below are satisfied:

$$(5) \qquad u=\sum_{i=1}^{n} \omega(u, v_i)u_i, \qquad \text{for } u \in U,$$

$$(6) \qquad v=\sum_{i=1}^{n} v_i\omega(u_i, v) \qquad \text{for } v \in V.$$

If $\{u_i\}$ and $\{v_i\}$ are B-bases of U and V respectively and $\omega(u_i, v_j)=\delta_{ij}1_B$, $\{u_i\}$ and $\{v_i\}$ are called *dual bases* of U and V; in this case $\{u_i\}$ and $\{v_i\}$ are, of course, dual sets of generators of U and V. Conversely, if $\{u_i\}$ and $\{v_i\}$ are dual sets of generators of U and V with respect to a B-bilinear form ω and if $\{u_i\}$ is a B-basis of U, then $\omega(u_i, v_j)=\delta_{ij}1_B$ and $\{v_i\}$ is a B-basis of V, that is, $\{u_i\}$ and $\{v_i\}$ are dual bases.

THEOREM 2.1. *Let ${}_BU_A$ be a B-A-bimodule and ${}_AV_B$ an A-B-bimodule. Then the following conditions are equivalent.*

6) Lemma 1.4 as well as its proof is a slight generalization of Müller [13, Hilfssatz 1].
6a) The first part of Lemma 1.5 is essentially the same as H. Bass [2, Lemma 1].

I. a) ${}_BU$ *is finitely generated and projective.*
b) ${}_AV_B \cong {}_A[\mathrm{Hom}_B({}_BU_A, {}_BB)]_B$.
II. a) V_B *is finitely generated and projective,*
b) ${}_BU_A \cong {}_B[\mathrm{Hom}_B({}_AV_B, B_B]_A$.
III. *There exists a natural A-isomorphism*

$${}_AV_B \otimes {}_BY \cong \mathrm{Hom}_B({}_BU_A, Y) \qquad \text{for } Y \in {}_B\mathfrak{M}.$$

IV. *There exists a B-bilinear form* ω *on* $U \times V$ *such that*

$$\omega(ua, v) = \omega(u, av) \qquad \text{for } a \in A,\ u \in U,\ v \in V, \tag{7}$$

and that there are dual sets of generators $\{u_i, i=1,\cdots, n\}$ *and* $\{v_i, i=1,\cdots, n\}$ *of* U *and* V *with respect to* ω.

PROOF. The equivalence I↔II is evident.

I→III. Assume I. Then by I a) the homomorphism

$$\lambda : \mathrm{Hom}_B({}_BU, {}_BB) \otimes Y \to \mathrm{Hom}_B({}_BU, Y)$$

defined by $[\lambda(f \otimes y)](u) = f(u)y$ for $f \in \mathrm{Hom}_B(U, {}_BB)$, $y \in Y$, $u \in U$, is a natural isomorphism by virtue of S. MacLane [9, p. 147]. It is easy to see that λ is a left A-isomorphism. Hence III holds in view of I b).

III→IV. Let

$$\mu(Y) : {}_AV_B \otimes {}_BY \cong \mathrm{Hom}_B({}_BU_A, Y), \qquad Y \in {}_B\mathfrak{M}$$

be a natural A-isomorphism. Then we have

$$\mu(B) : {}_AV_B \otimes {}_BB \cong \mathrm{Hom}_B({}_BU_A, {}_BB) \tag{8}$$

by setting $Y = {}_BB$. From the naturality of μ it follows that $\mu(B)$ is a right B-isomorphism if both sides of (8) are viewed as right B-modules as usual. Now set,

$$\omega(u, v) = [\mu(B)(v \otimes 1_B)](u), \qquad \text{for } u \in U,\ v \in V.$$

Then ω is additive with respect to $u \in U$ and $v \in V$, and we can easily prove that

$$\omega(bu, v) = b\omega(u, v), \ \ \omega(u, vb) = \omega(u, v)b$$

$$\omega(ua, v) = \omega(u, av)$$

where $a \in A$, $b \in B$, $u \in U$, $v \in V$.

If we set

$$\varphi_u(b) = bu \qquad \text{for } u \in U,\ b \in B,$$

then we have $\varphi_u \in \mathrm{Hom}_B({}_BB, {}_BU)$ and the diagram

$$\begin{array}{ccc} {}_AV_B \otimes {}_BB & \xrightarrow{\mu(B)} & \mathrm{Hom}({}_BU_A, {}_BB) \\ \Big\downarrow 1\otimes\varphi_u & & \Big\downarrow \mathrm{Hom}(1, \varphi_u) \\ {}_AV_B \otimes {}_BU & \xrightarrow{\mu(U)} & \mathrm{Hom}_B({}_BU_A, {}_BU) \end{array}$$

is commutative. Hence we have

$$(9) \qquad [(\mu(B)(v\otimes b))(u')]u = [\mu(U)(v\otimes bu)](u')$$

for u, $u'\in U$, $v\in V$, $b\in B$.

Let us set

$$C = [\mathrm{End}_B({}_BU)]^0 = [\mathrm{Hom}_B({}_BU, {}_BU)]^0.$$

Then, by the convention made in § 1, (9) can be written as follows:

$$\omega(u', vb)u = u'[\mu(U)(v\otimes bu)].$$

If we set $b=1_B$, we have then

$$(10) \qquad \omega(u', v)u = u'[\mu(U)(v\otimes u)],$$

where u, $u'\in U$, $v\in V$. If we denote by 1_C the identity element of C, then there are a finite number of elements $u_i\in U$, $v_i\in V$, $i=1,\cdots,n$ such that

$$\mu(U)^{-1}(1_C) = \sum_{i=1}^n v_i\otimes u_i,$$

since $\mu(U)$ is an isomorphism. Hence we have

$$u = u1_C = \sum_{i=1}^n u[\mu(U)(v_i\otimes u_i)],$$

and consequently by (10) we get

$$(11) \qquad u = \sum_{i=1}^n \omega(u, v_i)u_i \qquad \text{for } u\in U.$$

Now, let $v\in V$ and set

$$(12) \qquad v_0 = v - \sum_{i=1}^n v_i\omega(u_i, v).$$

Then by (11) we can prove that $\omega(u, v_0)=0$ for all $u\in U$. From the definition of ω it follows that $[\mu(B)(v_0\otimes 1_B)](u)=0$ for all $u\in U$. Since $\mu(B)$ is an isomorphism, we have $v_0=0$. Thus $\{u_i\}$ and $\{v_i\}$ are dual sets of generators of U and V with respect to ω.

IV→I. Assume IV. Then by Cartan-Eilenberg [3, p. 132] we see that ${}_BU$ and V_B are finitely generated and projective. Let us set

$$\Phi(f) = \sum_{i=1}^n v_if(u_i) \qquad \text{for } f\in\mathrm{Hom}_B({}_BU, {}_BB).$$

Then by (5) we have $f(u)=\omega(u, \Phi(f))$ for $u\in U$. Hence Φ is a one-to-one map from $\mathrm{Hom}_B({}_BU, {}_BB)$ into V. For $v_0\in V$ we have $\Phi(f_0)=v_0$ if we set $f_0(u)=\omega(u, v_0)$. Since it is easy to see that Φ is an A-B-homomorphism, statement I holds.

COROLLARY 2.2. *Suppose that condition I of Theorem 2.1 is satisfied, and set*

$$\omega(u, v)=[\tau(v)](u) \qquad \textit{for } u\in U,\ v\in V,$$

where τ is an A-B-isomorphism from ${}_AV_B$ onto $\mathrm{Hom}_B({}_BU_A, {}_BB)$. Then for any set of generators $\{u_1,\cdots, u_n\}$ of ${}_BU$, there exists a set of generators $\{v_1,\cdots, v_n\}$ of V_B such that $\{u_i\}$ and $\{v_i\}$ are dual sets of generators of U and V with respect to ω.

PROOF. From the proof of Theorem 2.1 it follows that there are $\{u_i\}$ and $\{v_i\}$ which are dual sets of generators of U and V with respect to ω. Let $\{u'_j|j=1,\cdots, m\}$ be any set of generators of ${}_BU$. Then there exist $b_{ij}\in B$ such that $u_i=\sum_j b_{ij}u'_j$. Set $v'_j=\sum_i v_ib_{ij}$. Then it is easy to see that $\{u'_j\}$ and $\{v'_j\}$ are dual sets of generators of U and V with respect to ω.

LEMMA 2.3. *Let U be a finitely generated, projective, left B-module and $V_B \cong[\mathrm{Hom}_B(U, {}_BB)]_B$. Then $[\mathrm{End}_B({}_BU)]^0$ and $\mathrm{End}_B(V_B)$ are isomorphic.*

LEMMA 2.4. *Let ${}_BU_A$ and ${}_AV_B$ be bimodules satisfying condition I of Theorem 2.1. Then, if ${}_AV$ is faithful, then the left A-module ${}_AV_B\otimes Y$ is faithful for any faithful left B-module Y.*

PROOF. Let $a_0\in A$ and $a_0\neq 0$. Then there is $v_0\in V$ such that $a_0v_0\neq 0$. Hence there is $u_0\in U$ such that $\omega(u_0, a_0v_0)\neq 0$. Since Y is faithful there is $y_0\in Y$ such that $\omega(u_0a_0, v_0)y_0\neq 0$. Let us set $\psi(u)=\omega(u, v_0)y_0$ for $u\in U$. Then $\psi\in\mathrm{Hom}_B({}_BU, Y)$ and $(a_0\psi)(u_0)\neq 0$. This proves Lemma 2.4 in view of Theorem 2.1.

§ 3. Adjoint pairs of functors

Througout this paper it will be assumed that S, S' are functors from ${}_A\mathfrak{M}$ to ${}_B\mathfrak{M}$ and T is a functor from ${}_B\mathfrak{M}$ to ${}_A\mathfrak{M}$.

In case S is a left adjoint of T and T has a right adjoint, that is, for some S' there are natural isomorphisms:

(13) $$\lambda(X, Y): \mathrm{Hom}_B(S(X), Y)\cong\mathrm{Hom}_A(X, T(Y)),$$

(14) $$\mu(X, Y): \mathrm{Hom}_A(T(Y), X)\cong\mathrm{Hom}_B(Y, S'(X)),$$

we shall say that $\{S, T\}$ is an *adjoint pair of functors*; in this case by S' we shall always mean a right adjoint of T.

For example, in case there is a ring homomorphism φ from A into B such that $\varphi(1_A)=1_B$, if we set $S(X)={}_BB_A\otimes X$, $T(Y)={}_AB_B\otimes Y$, $S'(X)=\mathrm{Hom}_A({}_AB_B, X)$ where we define ${}_AB$, B_A by $a*b=\varphi(a)b$, $b*a=b\varphi(a)$ for $b\in B$, $a\in A$, then $\{S, T\}$ is an adjoint pair of functors.

THEOREM 3.1. *$\{S, T\}$ is an adjoint pair of functors if and only if there are natural equivalences*

(15) $$S(X)\cong{}_BU_A\otimes X,\ T(Y)\cong{}_AV_B\otimes Y, \qquad X\in{}_A\mathfrak{M},\ Y\in{}_B\mathfrak{M},$$

with bimodules ${}_BU_A$ *and* ${}_AV_B$ *satisfying conditions* (16) *and* (17) *below:*

(16) $\qquad {}_BU$ *is finitely generated and projective,*

(17) $\qquad {}_AV_B \cong {}_A[\mathrm{Hom}_B({}_BU_A, {}_BB)]_B.$

REMARK. The conditions (16) and (17) are equivalent to conditions (16)′ and (17)′:

(16)′ $\qquad V_B$ is finitely generated and projective,

(17)′ $\qquad {}_BU_A \cong {}_B[\mathrm{Hom}_B({}_AV_B, B_B)]_A.$

PROOF OF THEOREM 3.1. Suppose that $\{S, T\}$ is an adjoint pair. Then we have natural equivalences (13) and (14). Let us set

(18) $$U = S({}_AA), \quad V = T({}_BB).$$

Then $U \in {}_B\mathfrak{M}$ and $V \in {}_A\mathfrak{M}$. For any element a of A we define $\varphi_a \in \mathrm{Hom}_A({}_AA, {}_AA)$ by $\varphi_a(x) = xa$, $x \in A$, and set

(19) $$ua = S(\varphi_a)u \qquad \text{for } u \in U,\ a \in A.$$

Then U becomes a right A-module, and moreover ${}_BU_A$ is a B-A-bimodule. Similarly, let us set

(20) $$vb = T(\psi_b)v \qquad \text{for } v \in V,\ b \in B,$$

where $\psi_b \in \mathrm{Hom}_B({}_BB, {}_BB)$ is defined by $\psi_b(y) = yb$ for $y \in B$. Then V becomes an A-B-bimodule.

From the natural isomorphism λ in (13) we obtain a natural isomorphism in $Y \in {}_B\mathfrak{M}$:

$$\mathrm{Hom}_B({}_BU_A, Y) \cong \mathrm{Hom}_A({}_AA, T(Y)).$$

Moreover, this isomorphism is a left A-isomorphism as is easily seen from the naturality of λ in $X \in {}_A\mathfrak{M}$. Thus we have a natural A-isomorphism

(21) $$T(Y) \cong \mathrm{Hom}_B({}_BU_A, Y) \qquad \text{for } Y \in {}_B\mathfrak{M}.$$

Similarly, from (14) we obtain a natural B-isomorphism

(22) $$S'(X) \cong \mathrm{Hom}_A({}_AV_B, X), \qquad \text{for } X \in {}_A\mathfrak{M}.$$

On the other hand, by [3, p. 28] we have a natural isomorphism

(23) $$\mathrm{Hom}_B({}_BU_A \otimes X, Y) = \mathrm{Hom}_A(X, \mathrm{Hom}_B({}_BU_A, Y))$$

for $X \in {}_A\mathfrak{M}$, $Y \in {}_B\mathfrak{M}$. As is proved by D. M. Kan [5], a left adjoint of a functor, if it exists, is determined uniquely up to a natural equivalence. Therefore by (21) we obtain a natural B-isomorphism

(24) $$S(X) \cong {}_B U_A \otimes X, \qquad X \in {}_A\mathfrak{M}.$$

Similarly, from (22) we get a natural A-isomorphism

(25) $$T(Y) \cong {}_A V_B \otimes Y, \qquad Y \in {}_B\mathfrak{M}.$$

In view of Theorem 2.1, the proof of the "only if" part of the theorem is now completed by (21) and (25).

The "if" part of Theorem 3.1 is stated in the following theorem more explicity.

THEOREM 3.2. *Suppose that S, S' and T are defined by*

$$S(X) = {}_B U_A \otimes X, \; S'(X) = \mathrm{Hom}_A({}_A V_B, X), \qquad \textit{for } X \in {}_A\mathfrak{M},$$

$$T(Y) = {}_A V_B \otimes Y, \qquad \textit{for } Y \in {}_B\mathfrak{M},$$

where ${}_B U_A$ and ${}_A V_B$ are bimodules satisfying conditions (16) *and* (17) *of Theorem* 3.1. *Let us set*

(26) $$\omega(u, v) = [\tau(v)](u) \qquad \textit{for } u \in U, \; v \in V$$

where τ is an A-B-isomorphism: ${}_A V_B \cong {}_A[\mathrm{Hom}_B({}_B U_A, {}_B B)]_B$. Let $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_n\}$ be dual sets of generators of ${}_B U$ and V_B with respect to a B-bilinear form ω.

Then the homomorphisms

$$\lambda: \mathrm{Hom}_B(S(X), Y) \to \mathrm{Hom}_A(X, T(Y)),$$

$$\mu: \mathrm{Hom}_A(T(Y), X) \to \mathrm{Hom}_B(Y, S'(X))$$

defined by

(27) $$[\lambda(g)](x) = \sum_{i=1}^{n} v_i \otimes g(u_i \otimes x), \qquad \textit{for } x \in X,$$

(28) $$[\mu(f)(y)](v) = f(v \otimes y), \qquad \textit{for } y \in Y, \; v \in V,$$

are isomorphisms which are natural in $X \in {}_A\mathfrak{M}$ and $Y \in {}_B\mathfrak{M}$.

PROOF. As is well known, the homomorphisms

$$\Phi_0: \mathrm{Hom}_B({}_B U_A \otimes X, Y) \to \mathrm{Hom}_A(X, \mathrm{Hom}_B({}_B U_A, Y))$$

$$\Phi_1: \mathrm{Hom}_B({}_B U_A, {}_B B) \otimes {}_B Y \to \mathrm{Hom}_B({}_B U_A, Y)$$

defined by

(29) $$[\Phi_0(g)(x)](u) = g(u \otimes x),$$

(30) $$[\Phi_1(h \otimes y)](u) = h(u)y,$$

where $x \in X$, $y \in Y$, $u \in U$, $g \in \mathrm{Hom}_B({}_B U_A \otimes X, Y)$, $h \in \mathrm{Hom}_B({}_B U_A, {}_B B)$, are natural isomorphisms; the isomorphism of Φ_1 follows from (16).

Let us set

(31) $$\Phi_2=\Phi_1\circ(\tau\otimes 1):\ {}_AV_B\otimes Y\to \mathrm{Hom}_B({}_BU_A,\ Y)$$

(32) $$\lambda=\mathrm{Hom}(1,\ \Phi_2^{-1})\circ\Phi_0:\ \mathrm{Hom}_B(S(X),\ Y)\to\mathrm{Hom}_A(X,\ T(Y)).$$

Then Φ_2 is a natural isomorphism in Y and λ is an isomorphism which is natural in X and Y.

Since it follows from (30) that

(33) $$[\Phi_2(v\otimes y)](u)=[\tau(v)(u)]y=\omega(u,\ v)y,$$

we have for $g\in\mathrm{Hom}_B(S(X),\ Y)$

$$[\Phi_2(\sum_{i=1}^{n}v_i\otimes g(u_i\otimes x))](u)=\sum_{i=1}^{n}\omega(u,\ v_i)g(u_i\otimes x)$$

$$=g(\sum_{i=1}^{n}\omega(u,\ v_i)u_i\otimes x)=g(u\otimes x);$$

we have used here the property (5) of dual sets of generators (cf. § 2).

Hence by (29) we have

$$\Phi_2(\sum_{i=1}^{n}v_i\otimes g(u_i\otimes x))=\Phi_0(g)(x),$$

which shows, in view of (32), that

(27) $$\lambda(g)(x)=\sum_{i=1}^{n}v_i\otimes g(u_i\otimes x).$$

Since it is obvious that μ defined by (28) is a natural isomorphism, the proof of Theorem 3. 2 is completed.

COROLLARY 3. 3. *Under the same assumptions as in Theorem* 3. 2, *the natural transformations*[7]

$$\alpha_0(X):\ X\to TS(X),\qquad \alpha_1(X):\ TS'(X)\to X,$$

$$\beta_0(Y):\ Y\to S'T(Y),\qquad \beta_1(Y):\ ST(Y)\to Y$$

defined by

$$\alpha_0(X)=\lambda(1_{S(X)}),\qquad \alpha_1(X)=\mu^{-1}(1_{S'(X)}),$$

$$\beta_0(Y)=\mu(1_{T(Y)}),\qquad \beta_1(Y)=\lambda^{-1}(1_{T(Y)})$$ [8]

are expressed as follows:

$$\alpha_0(X)(x)=\sum_{i=1}^{n}v_i\otimes u_i\otimes x,\qquad \text{for } x\in X,$$

$$\alpha_1(X)(v\otimes g)=g(v),\qquad \text{for } v\in V,\ g\in\mathrm{Hom}_A({}_AV,\ X),$$

7) Cf. § 6 below for the significance of these transformations.
8) We denote by 1_X the identity map from X onto itself.

$$[\beta_0(Y)(y)](v)=v\otimes y, \qquad \text{for } y\in Y,\ v\in V,$$

$$\beta_1(Y)(u\otimes v\otimes y)=\omega(u, v)y, \qquad \text{for } u\in U,\ v\in Y,\ y\in Y.$$

PROOF. The expressions for $\alpha_0(X)$ and $\beta_0(Y)$ are obvious from Theorem 3.2. We shall consider

$$\lambda^{-1}=\Phi_0^{-1}\circ[\mathrm{Hom}\,(1, \Phi_2^{-1})]^{-1}=\Phi_0^{-1}\circ\mathrm{Hom}\,(1, \Phi_2)$$

with the notations used in the proof of Theorem 3.2. If we set

$$g=[\mathrm{Hom}\,(1, \Phi_2)](1_{T(Y)}),$$

we have

$$[\Phi_0^{-1}(g)](u\otimes x)=g(x)(u) \qquad \text{by (29)},$$

$$[g(v\otimes y)](u)=\omega(u, v)y \qquad \text{by (33)}.$$

By setting $x=v\otimes y$ we then obtain

$$[\Phi_0^{-1}(g)](u\otimes v\otimes y)=[g(v\otimes y)](u)=\omega(u, v)y.$$

This shows that $\beta_1(Y)(u\otimes v\otimes y)=\omega(u, v)y$.

Next, we shall consider $\alpha_1(X)$. For $v\in V$, $f\in\mathrm{Hom}_A({}_AV_B, X)$ we have $\alpha_1(X)(v\otimes f)=f(v)$ by (28). Thus the proof of Corollary 3.3 is completed.

§ 4. Quasi-strongly adjoint pairs

An A-B-bimodule ${}_AW_B$ will be said to be *similar* to an A-B-bimodule ${}_AW'_B$ if ${}_AW_B$ is A-B-isomorphic to a direct summand of a finite direct sum of copies of ${}_AW'_B$ and if ${}_AW'_B$ is A-B-isomorphic to a direct summand of a finite direct sum of copies of ${}_AW_B$; we write ${}_AW_B\sim{}_AW'_B$ in this case. ${}_AW_B$ is similar to ${}_AW'_B$ if and only if there exist $\varphi_i, \varphi'_j\in\mathrm{Hom}_{(A,B)}({}_AW_B, {}_AW'_B)$, $\psi_i, \psi'_j\in\mathrm{Hom}_{(A,B)}({}_AW'_B, {}_AW_B)$, $i=1,\cdots,n$; $j=1,\cdots,m$, such that

$$\sum_{i=1}^{n}\psi_i\circ\varphi_i=1_W, \qquad \sum_{j=1}^{m}\varphi'_j\circ\psi'_j=1_{W'}.$$

The notion "similar" is likewise defined for one-sided modules[9].

For functors $S, S'\colon {}_A\mathfrak{M}\to{}_B\mathfrak{M}$ we shall say that S is *similar* to S', if for some positive integers r and s there are natural transformations

$$S(X)\xrightarrow{\varphi(X)}\sum_{i=1}^{r}\oplus S'(X)\xrightarrow{\psi(X)}S(X)$$

$$S'(X)\xrightarrow{\varphi'(X)}\sum_{j=1}^{s}\oplus S(X)\xrightarrow{\psi'(X)}S'(X)$$

9) A left A-module is similar to ${}_AA$ if and only if it is a finitely generated, projective generator.

such that $\psi(X)\circ\varphi(X)=1_{S(X)}$ and $\varphi'(X)\circ\psi'(X)=1_{S'(X)}$.

In case S is naturally equivalent to a functor defined as ${}_BU_A\otimes_A X$ with a B-A-bimodule ${}_BU_A$, S' is similar to S if and only if S' is naturally equivalent to a functor defined as ${}_BU'_A\otimes_A X$ with a B-A-bimodule ${}_BU'_A$ which is similar to ${}_BU_A$.

Let S_0 be a category-isomorphism from ${}_A\mathfrak{M}$ to itself and T_0 a category-isomorphism from ${}_B\mathfrak{M}$ to itself; let ${}_AM_A$, ${}_AM_A^{-1}$, ${}_BN_B$, ${}_BN_B^{-1}$ be bimodules such that

$$S_0(X)\cong {}_AM_A\otimes X,\ S_0^{-1}(X)\cong {}_AM_A^{-1}\otimes X,\qquad X\in {}_A\mathfrak{M},\tag{34}$$

$$T_0(Y)\cong {}_BN_B\otimes Y,\ T_0^{-1}(Y)\cong {}_BN_B^{-1}\otimes Y,\qquad Y\in {}_B\mathfrak{M}.\tag{35}$$

These notations shall be retained throughout this paper.

Let $\{S, T\}$ be an adjoint pair of functors such that T has a right adjoint S'. Now let us consider the following requirements for S':

1) $S'\cong S$,
2) $S'\cong T_0^{-1}SS_0$,
3) S' is similar to S,
4) S' is similar to $T_0^{-1}SS_0$.

According as 1), 2), 3) or 4) holds, we shall say that $\{S, T\}$ is a *strongly adjoint pair*, an (S_0, T_0)-*strongly adjoint pair*, a *quasi-strongly adjoint pair* or an (S_0, T_0)-*quasi-strongly adjoint pair.*

We shall now establish the following theorem.

THEOREM 4.1. *$\{S, T\}$ is an (S_0, T_0)-quasi-strongly adjoint pair if and only if there are natural equivalences:*

$$S(X)\cong {}_BU_A\otimes X,\ T(Y)\cong {}_AV_B\otimes Y,\ S'(X)\cong {}_BU'_A\otimes X\tag{36}$$

with bimodules ${}_BU_A$, ${}_AV_B$ and ${}_BU'_A$ satisfying conditions (37) *to* (40):

$$\text{(37)}\qquad {}_AV \textit{ and } V_B \textit{ are finitely generated and projective,}$$

$${}_BU_A\cong \mathrm{Hom}_B({}_AV_B, B_B),\tag{38}$$

$${}_BU'_A\cong \mathrm{Hom}_A({}_AV_B, {}_AA),\tag{39}$$

$${}_BU'_A\sim {}_BN_B^{-1}\otimes {}_BU_A\otimes {}_AM_A.\tag{40}$$

If the condition (40) *is replaced by* (40)′, (40)″, *or* (40)‴ *below, we have a necessary and sufficient condition for $\{S, T\}$ to be a strongly adjoint pair, an (S_0, T_0)-strongly adjoint pair, or a quasi-strongly adjoint pair:*

$${}_BU'_A\cong {}_BU_A,\tag{40$'$}$$

$${}_BU'_A\cong {}_BN_B^{-1}\otimes {}_BU_A\otimes {}_AM_A,\tag{40$''$}$$

$${}_BU'_A\sim {}_BU_A.\tag{40$'''$}$$

PROOF. Suppose that $\{S, T\}$ is an (S_0, T_0)-quasi-strongly adjoint pair. Then by Theorem 3.1 there are bimodules ${}_BU_A$ and ${}_AV_B$ such that (38) and the part of (37) for V_B hold. Since $T_0^{-1}SS_0(X)\cong {}_BN_B^{-1}\otimes {}_BU_A\otimes {}_AM_A\otimes {}_AX$, by the remark made at the beginning of this section there is a B-A-bimodule ${}_BU'_B$ such that $S'(X)\cong {}_BU'_A\otimes X$ and ${}_BU'_A\sim {}_BN_B^{-1}\otimes_B U_A\otimes_A M_A$. From the form of the functor S' it follows that S' has a right adjoint, and hence $\{T, S'\}$ is also an adjoint pair. Now Theorem 3.1 is applicable; thus ${}_AV$ is finitely generated and projective, and (39) holds. Therefore conditions (36) to (40) hold. Conversely, suppose that conditions (36) to (40) hold. Then $\{S, T\}$ and $\{T, S'\}$ are adjoint pairs by Theorem 3.1. Thus the first part of the theorem is proved. The second part is easy to see.

THEOREM 4.2. *$\{S, T\}$ and $\{T, S'\}$ are adjoint pairs if and only if there are bimodules ${}_BU_A$, ${}_AV_B$, and ${}_BU'_A$ satisfying conditions* (36) *to* (39) *in Theorem* 4.1. *The conditions* (37) *to* (39) *are satisfied if and only if there are an A-bilinear form ω_A on ${}_AV\times U'_A$ and a B-bilinear form ω_B on ${}_BU\times V_B$ such that*

$$(41)\qquad \omega_A(vb, u')=\omega_A(v, bu') \qquad \text{for } b\in B,\ v\in V,\ u'\in U',$$

$$(42)\qquad \omega_B(ua, v)=\omega_B(u, av) \qquad \text{for } a\in A,\ u\in U,\ v\in V,$$

and such that there are dual sets of generators $\{v'_j,\ j=1,\cdots,m\}$ and $\{u'_j,\ j=1,\cdots,m\}$ of ${}_AV$ and U'_A with respect to ω_A, and dual sets of generators $\{u_i,\ i=1,\cdots,n\}$ and $\{v_i,\ i=1,\cdots,n\}$ of ${}_BU$ and V_B with respect to ω_B.

The first part of Theorem 4.2 is contained in the proof of Theorem 4.1 and the second is a direct consequence of Theorem 2.1.

THEOREM 4.3. *Let us set*

$$\tilde{S}(X)=X\otimes_A V_B,\ \tilde{T}(Y)=Y\otimes_B U_A,\ \tilde{T}'(Y)=Y\otimes_B U'_A,$$

where $X\in\mathfrak{M}_A$, $Y\in\mathfrak{M}_B$. If the bimodules ${}_BU_A$, ${}_BU'_A$ and ${}_AV_B$ satisfy conditions (37) *to* (39), *$\{\tilde{S}, \tilde{T}\}$ and $\{\tilde{T}', \tilde{S}\}$ are adjoint pairs, and conversely.*

Proof is obvious.

§ 5. Frobenius and quasi-Frobenius extensions

Throughout this section B is assumed to be a subring of A.

We shall say that A is an (S_0, T_0)-*Frobenius* (resp. (S_0, T_0)-*quasi-Frobenius*) *extension* of B if conditions (43) and (44) (resp. (43) and (45)) below are satisfied:

$$(43)\qquad A_B \text{ is finitely generated and projective,}$$

$$(44)\qquad {}_BA_A\cong {}_BN_B^{-1}\otimes_B[\mathrm{Hom}_B({}_AA_B, B_B)]_A\otimes_A M_A,$$

$$(45)\qquad {}_BA_A\sim {}_BN_B^{-1}\otimes_B[\mathrm{Hom}_B({}_AA_B, B_B)]_A\otimes_A M_A.$$

Then we have

THEOREM 5.1. *Let us set $T(Y)={}_AA_B\otimes_B Y$, $Y\in{}_B\mathfrak{M}$. Then the following statements are equivalent.*

1) *A is an (S_0, T_0)-Frobenius (resp. (S_0, T_0)-quasi-Frobenius) extension of B.*

2) *$\{S, T\}$ is an (S_0, T_0)-strongly (resp. (S_0, T_0)-quasi-strongly) adjoint pair for some functor S: ${}_A\mathfrak{M}\rightarrow{}_B\mathfrak{M}$.*

3) *There is a natural isomorphism:*

$$\mathrm{Hom}_B({}_BU_A\otimes X, Y)\cong \mathrm{Hom}_A(X, T(Y)) \qquad \textit{for } X\in {}_A\mathfrak{M},\ Y\in {}_B\mathfrak{M}$$

with a B-A-*bimodule* ${}_BU_A$ *such that* ${}_BU_A\cong$ (*resp.* $\sim$) ${}_BN_B\otimes {}_BA_A\otimes {}_AM_A^{-1}$.

PROOF. If we set $S'(X)={}_BA_A\otimes X$ $(X\in {}_A\mathfrak{M})$, then $\{T, S'\}$ is always an adjoint pair, since $\mathrm{Hom}_A({}_AA_B, X)\cong {}_BA_A\otimes X$. Hence statements 2) and 3) are equivalent. The equivalence of 1) and 2) follows readily from Theorem 4.1.

LEMMA 5.2. *The conditions* (43) *and* (44) (*resp.* (43) *and* (45)) *are equivalent to conditions* (43)′ *and* (44)′ (*resp.* (43)′ *and* (45)′) *below:*

(43)′ ${}_BA$ *is finitely generated and projective,*

(44)′ ${}_AA_B\cong {}_AM_A\otimes {}_A[\mathrm{Hom}_B({}_BA_A, {}_BB)]_B\otimes {}_BN_B^{-1}$,

(45)′ ${}_AA_B\sim {}_AM_A\otimes {}_A[\mathrm{Hom}_B({}_BA_A, {}_BB)]_B\otimes {}_BN_B^{-1}$.

PROOF. Assume (43) and (44). Then we have (43)′ and

$$\begin{aligned}\mathrm{Hom}_B({}_BA_A, {}_BB)&\cong \mathrm{Hom}_B({}_BN_B^{-1}\otimes[\mathrm{Hom}_B({}_AA_B, B_B)]\otimes {}_AM_A, {}_BB)\\ &\cong \mathrm{Hom}_B([\mathrm{Hom}_B({}_AA_B, B_B)]\otimes {}_AM_A, {}_BN_B)\\ &\cong \mathrm{Hom}_A({}_AM_A, {}_AA_B\otimes {}_BN_B)\cong {}_AM_A^{-1}\otimes {}_AA_B\otimes {}_BN_B.\end{aligned}$$

The remaining parts are similarly proved.

We shall next consider the case where S_0 and T_0 are induced by automorphisms α and β of A and B respectively; that is,

(46) $$S_0(X)=(\alpha, X),\ T_0(Y)=(\beta, Y), \qquad X\in {}_A\mathfrak{M},\ Y\in {}_B\mathfrak{M}.$$

Here for a left A-module X we denote by (α, X) the left A-module which coincides with X as additive groups and has a new left A-module structure defined by $a*x=\alpha(a)x$ for $a\in A$, $x\in X$; for a right A-module X the right A-module (X, α) is similarly defined.

In case (46) holds we have

(47) $${}_AM_A\cong {}_A(\alpha, {}_AA)_A\cong {}_A(A_A, \alpha^{-1})_A,\ {}_BN_B\cong {}_B(\beta, {}_BB)_B\cong {}_B(B_B, \beta^{-1})_B$$

and hence conditions (44) and (44)′ are stated as follows:

(48) $${}_B[\beta, ({}_BA_A, \alpha)]_A\cong {}_B[\mathrm{Hom}_B({}_AA_B, B_B)]_A,$$

(48)′ $${}_A[\alpha^{-1}, ({}_AA_B, \beta^{-1})]_B\cong {}_A[\mathrm{Hom}_B({}_BA_A, {}_BB)]_B;$$

if we replace $\cong$ by $\sim$, we have the conditions corresponding to (45) and (45)′.

In case (46) holds we shall speak of an (α, β)-*Frobenius* (resp. (α, β)-*quasi-Frobenius*) *extension* instead of an (S_0, T_0)-Frobenius (resp. (S_0, T_0)-quasi-Frobenius) extension. In the special case that $\alpha=1$, $\beta=1$ we shall simply speak of a Frobenius (resp. quasi-Frobenius) extension; this definition agrees with those given by F.

10) More precisely, a quasi-Frobenius extension in our sense is a two-sided quasi-Frobenius extension in the sense of Müller. One-sided quasi-Frobenius extensions in his sense can also be treated in the framework of adjoint pairs of functors as is easily seen.

Kasch [7] and by B. Müller [13][10]. A β-Frobenius extension in the sense of Nakayama-Tsuzuku [14] is nothing else a $(1, \beta)$-Frobenius extension.

LEMMA 5.3. *A is an (α, β)-Frobenius extension of B if and only if there is a map*

$$\omega: A \times A \to B$$

such that ω is additive with respect to each variable and

$$\omega(\beta(b)u, v) = b\omega(u, v),$$

$$\omega(u, vb) = \omega(u, v)b, \qquad u, v \in A, \ b \in B, \ a \in A,$$

$$\omega(u\alpha(a), b) = \omega(u, av),$$

and there exist two sets $\{u_i, i=1, \cdots, n\}$, $\{v_i, i=1, \cdots, n\}$ of elements of A satisfying the conditions below

$$u = \sum_{i=1}^{n} \beta[(\omega(u, v_i)]u_i, \qquad u \in A,$$

$$v = \sum_{i=1}^{n} v_i \omega(u_i, v), \qquad v \in A.$$

Proof is obvious from Theorem 4.2 in view of (47).

An analogous theorem corresponding to (α, β)-quasi-Frobenius extensions can be obtained.

An (α, β)-Frobenius (resp. (α, β)-quasi-Frobenius) extension A of B is an (α', β')-Frobenius (resp. (α', β')-quasi-Frobenius) extension of B if there is an element a_0 of A such that a_0 has an inverse a_0^{-1} and $a_0^{-1}\beta'(b)a_0 = \alpha'\alpha^{-1}(\beta(b))$ for all $b \in B$, since the latter condition holds if and only if $(\beta, ({}_BA_A, \alpha)) \cong (\beta', ({}_BA_A, \alpha'))$. In particular, in case $\alpha | B = \beta$, a $(1, \beta)$-Frobenius extension A of B is an $(\alpha^{-1}, 1)$-Frobenius extension of B.

In case B is contained in the center of A, any (α, β)-quasi-Frobenius extension A of B is always a quasi-Frobenius extension of B. Because for some integer $m > 0$ there are B-A-homomorphisms φ, ψ:

$$(\beta, ({}_BA_A, \alpha)) \xrightarrow{\varphi} \sum_{i=1}^{m} \oplus \operatorname{Hom}({}_AA_B, B_B) \xrightarrow{\psi} (\beta, ({}_BA_A, \alpha))$$

such that $\psi \circ \varphi =$ identity, and if we set $\varphi(1) = (u_1, \cdots, u_m)$ then for $b \in B$ we have

$$\varphi(\beta(b)) = (bu_1, \cdots bu_m) = (u_1 b, \cdots, u_m b) = \varphi(\alpha(b))$$

and hence $\beta(b) = \alpha(b)$.

§ 6. Relatively projective and injective modules

Let $\{S, T\}$ be an adjoint pair and S' a right adjoint of T; then there are natural isomorphisms:

(13) $$\lambda(X, Y): \mathrm{Hom}_B(S(X), Y) \cong \mathrm{Hom}_A(X, T(Y)),$$

(14) $$\mu(Y, X): \mathrm{Hom}_A(T(Y), X) \cong \mathrm{Hom}_B(Y, S'(X)).$$

Let us set

(49) $$\alpha_0(X) = [\lambda(X, S(X))](1_{S(X)}): X \to TS(X),$$

(50) $$\alpha_1(X) = [\mu(S'(X), X)]^{-1}(1_{S'(X)}): TS'(X) \to X,$$

(51) $$\beta_0(Y) = [\mu(Y, T(Y))](1_{T(Y)}): Y \to S'T(Y),$$

(52) $$\beta_1(Y) = [\lambda(T(Y), Y)]^{-1}(1_{T(Y)}): ST(Y) \to Y.$$

Then α_0, α_1, β_0, β_1 are natural transformations between functors and we have the following factorizations:

(53) $$f = T(\lambda^{-1}(f)) \circ \alpha_0(X) \qquad \text{for } f \in \mathrm{Hom}_A(X, T(Y)),$$

(54) $$f' = \alpha_1(X) \circ T(\mu(f')) \qquad \text{for } f' \in \mathrm{Hom}_A(T(Y), X),$$

(55) $$g = S'(\mu^{-1}(g)) \circ \beta_0(Y) \qquad \text{for } g \in \mathrm{Hom}_B(Y, S'(X)),$$

(56) $$g' = \beta_1(Y) \circ S(\lambda(g')) \qquad \text{for } g' \in \mathrm{Hom}_B(S(X), Y);$$

these are proved by D. M. Kan [5].

A left A-module X will be called *S'-projective* if for any $f \in \mathrm{Hom}_A(X, X')$, $g \in \mathrm{Hom}_A(X'', X')$, $X', X'' \in {}_A\mathfrak{M}$ such that $S'(g) \circ k = 1_{S'(X')}$ for some $k \in \mathrm{Hom}_B(S'(X'), S'(X''))$, there exists $h \in \mathrm{Hom}_A(X, X'')$ such that $f = g \circ h$. A left A-module X will be called *S-injective* if for any $f \in \mathrm{Hom}_A(X', X)$, $g \in \mathrm{Hom}_A(X', X'')$, $X', X'' \in {}_A\mathfrak{M}$ such that $k \circ S(g) = 1_{S(X')}$ for some $k \in \mathrm{Hom}_B(S(X''), S(X'))$, there exists $h \in \mathrm{Hom}_A(X'', X)$ such that $f = h \circ g$.

Then we have the following lemmas.

LEMMA 6.1. *$T(Y)$ is S'-projective for $Y \in {}_B\mathfrak{M}$.*

PROOF. Let $f \in \mathrm{Hom}_A(T(Y), X)$, $g \in \mathrm{Hom}_A(X', X)$ and suppose that $S'(g) \circ k = 1_{S'(X)}$ for some $k \in \mathrm{Hom}_B(S'(X), S'(X'))$. Let us set $h_0 = k \circ S'(f) \circ \beta_0(Y)$. Then $h_0 = S'(\mu^{-1}(h_0)) \circ \beta_0(Y)$ by (55). Hence $S'(f) \circ \beta_0(Y) = S'(g) \circ h_0 = S'(g \circ h) \circ \beta_0(Y)$ where we set $h = \mu^{-1}(h_0)$. Again by (55) we have $f = g \circ h$.

LEMMA 6.2. *If a left A-module X is S'-projective, so is a direct summand of ${}_AX$.*

Proof is obvious.

LEMMA 6.3. *A left A-module X is S'-projective if and only if X is A-isomorphic to a direct summand of $TS'(X)$.*

PROOF. By (55) we have

$$1_{S'(X)} = S'(\mu^{-1}(1_{S'(X)})) \circ \beta_0(S'(X)) = S'(\alpha_1(X)) \circ \beta_0(S'(X)).$$

If X is S'-projective, then there exists $\rho \in \mathrm{Hom}_A(X, TS'(X))$ such that $1_X = \alpha_1(X) \circ \rho$, and hence the "only if" part is proved. The "if" part is obvious by Lemmas 6.1 and 6.2.

Dually to these lemmas we have

LEMMA 6.1′. *$T(Y)$ is S-injective for $Y \in {}_B\mathfrak{M}$.*

LEMMA 6.2′. *If a left A-module X is S-injective, so is a direct summand of ${}_AX$.*

LEMMA 6.3′. *A left A-module X is S-injective if and only if X is A-isomorphic to a direct summand of $TS(X)$.*

Now we shall define the *transfer* homomorphism

$$t\colon \mathrm{Hom}_B(S(X), S'(X')) \to \mathrm{Hom}_A(X, X') \tag{57}$$

by

$$t(g) = \alpha_1(X') \circ T(g) \circ \alpha_0(X), \qquad \text{for } g \in \mathrm{Hom}_B(S(X), S'(X')), \tag{58}$$

where $X, X' \in {}_A\mathfrak{M}$.

Then it is easy to see that

$$t(S'(h) \circ g \circ S(f)) = h \circ t(g) \circ f \tag{59}$$

for $f \in \mathrm{Hom}_A(X'', X)$, $h \in \mathrm{Hom}_A(X', X''')$.

LEMMA 6.4. *For any left B-module Y we have $1_{T(Y)} = t(\beta_0(Y) \circ \beta_1(Y))$.*

PROOF. The diagram

$$\begin{array}{ccc}
\mathrm{Hom}_A(T(Y), T(Y)) & \xrightarrow{\ \mu\ } & \mathrm{Hom}_B(Y, S'T(Y)) \\
\Big\downarrow {\scriptstyle \mathrm{Hom}(T(\beta_1(Y)), 1)} & & \Big\downarrow {\scriptstyle \mathrm{Hom}(\beta_1(Y), 1)} \\
\mathrm{Hom}_A(TST(Y), T(Y)) & \xrightarrow{\ \mu\ } & \mathrm{Hom}_B(ST(Y), S'T(Y))
\end{array}$$

is commutative, and hence we have

$$\begin{aligned}
\beta_0(Y) \circ \beta_1(Y) &= [\mathrm{Hom}(\beta_1(Y), 1) \circ \mu](1_{T(Y)}) \\
&= [\mu \circ \mathrm{Hom}(T(\beta_1(Y)), 1)](1_{T(Y)}) \\
&= \mu(T(\beta_1(Y))).
\end{aligned}$$

On the other hand, by (53) we have the factorization

$$1_{T(Y)} = T(\lambda^{-1}(1_{T(Y)})) \circ \alpha_0(T(Y)) = T(\beta_1(Y)) \circ \alpha_0(T(Y)),$$

and by (54)

$$T(\beta_1(Y)) = \alpha_1(T(Y)) \circ T(\mu(T(\beta_1(Y)))).$$

Thus Lemma 6.4 is proved.

Now we are in a position to establish the following theorem.

THEOREM 6.5. *For a left A-module X the following statements are equivalent.*

(a) *X is S'-projective.*

(b) *X is S-injective.*

(c) *There exists $h \in \mathrm{Hom}_B(S(X), S'(X))$ such that $1_X = t(h)$, where 1_X is the identity* *map of X onto itself.*

PROOF. (a)→(c). Assume (a). Then there exists $\rho \in \mathrm{Hom}_A(X, TS'(X))$ such that $1_X = \alpha_1(X) \circ \rho$ (cf. the proof of Lemma 6.3.). Since the diagrams

$$\begin{tikzcd} TS'(X) \arrow[r, "\alpha_1(X)"] & X \\ TS'T(Y) \arrow[u, "TS'(\sigma)"] \arrow[r, "\alpha_1(T(Y))"'] & T(Y) \arrow[u, "\sigma"'] \end{tikzcd}, \qquad \begin{tikzcd} X \arrow[r, "\alpha_0(X)"] \arrow[d, "\rho"] & TS(X) \arrow[d, "TS(\rho)"] \\ T(Y) \arrow[r, "\alpha_0(T(Y))"] & TST(Y) \end{tikzcd}$$

are commutative where we set $\sigma = \alpha_1(X)$, $Y = S'(X)$. By Lemma 6.4 we have

$$\begin{aligned} 1_X &= \sigma \circ 1_{T(Y)} \circ \rho \\ &= \sigma \circ \alpha_1(T(Y)) \circ T(\beta_0(Y) \circ \beta_1(Y)) \circ \alpha_0(T(Y)) \circ \rho \\ &= \alpha_1(X) \circ TS'(\sigma) \circ T(\beta_0(Y) \circ \beta_1(Y)) \circ TS(\rho) \circ \alpha_0(X). \end{aligned}$$

Hence, if we set $h = S'(\sigma) \circ \beta_0(Y) \circ \beta_1(Y) \circ S(\rho)$, we get $h \in \mathrm{Hom}_B(S(X), S'(X))$ and

$$1_X = \alpha_1(X) \circ T(h) \circ \alpha_0(X) = t(h).$$

Thus (c) holds.

(c)→(a). If (c) holds, then we have $1_X = \alpha_1(X) \circ \rho$ where $\rho = T(h) \circ \alpha_0(X) \in \mathrm{Hom}_A(X, TS'(X))$. This shows that X is A-isomorphic to a direct summand of $TS'(X)$, and hence by Lemma 6.3 we see that X is S'-projective.

The implications (b)→(c), (c)→(b) are proved similarly. Thus Theorem 6.5 is proved.

LEMMA 6.6. *If a left A-module X is projective, so is $S(X)$. More generally, this holds in case S is a left adjoint of an exact functor T.*

LEMMA 6.6′. *If a left A-module X is injective, so is $S'(X)$. More generally, this holds in case S' is a right adjoint of an exact functor T.*

We shall prove Lemma 6.6′. Let g: $Y \to Y'$ be a B-monomorphism. Then $T(g)$: $T(Y) \to T(Y')$ is an A-monomorphism. Let $f \in \mathrm{Hom}_B(Y, S'(X))$. Then by the injectivity of X there is $h \in \mathrm{Hom}_A(T(Y'), X)$ such that $\mu^{-1}(f) = h \circ T(g)$. On the other hand, $f = S'(\mu^{-1}(f)) \circ \beta_0(Y)$ by (55). Hence $f = S'(h) \circ S'T(g) \circ \beta_0(Y) = S'(h) \circ \beta_0(Y') \circ g$. This proves Lemma 6.6′.

It is to be noted that the arguments so far are of such a nature that they may be applied to adjoint pairs of functors for abelian categories as well. Lemmas 6.1 to 6.3 (resp. 6.1′ to 6.3′) are valid under the condition that S' (resp. S) is a right (resp. left) adjoint of T.

Now we shall apply Theorem 6.5 to quasi-strongly adjoint pairs.

LEMMA 6.7. *If $\{S, T\}$ is an (S_0, T_0)-quasi-strongly adjoint pair, each of statements* (a)′ *and* (b)′ *below for $X \in {}_A\mathfrak{M}$ is equivalent to each of* (a) *and* (b) *of Theorem* 6.5.

(a)′ *$S_0(X)$ is S-projective.*

(b)′ $S_0^{-1}(X)$ *is* S'*-injective.*

PROOF. S' is similar to $T_0^{-1}SS_0$. Hence X is S'-projective if and only if X is $T_0^{-1}SS_0$-projective. Since T_0^{-1} is a category-isomorphism, X is $T_0^{-1}SS_0$-projective if and only if X is SS_0-projective, and the latter statement is equivalent to (a)′. Thus (a) and (a)′ are equivalent. Similarly (b) and (b)′ are equivalent.

By using the results of § 3 we can express the transfer homomorphism by means of dual sets of generators.

LEMMA 6.8. *Let* ${}_BU_A$, ${}_BU'_A$, ${}_AV_B$ *be bimodules satisfying conditions* (36) *to* (39) *of Theorem* 4.1. *Let us set*

$$\omega_A(v, u')=[\sigma(u')](v), \qquad u'\in U',\ v\in V,$$

$$\omega_B(u, v)=[\tau(v)](u), \qquad u\in U,\ v\in V,$$

where $\sigma\colon {}_BU'_A\cong {}_B[\mathrm{Hom}_A({}_AV_B, {}_AA)]_A$, $\tau\colon {}_AV_B\cong {}_A[\mathrm{Hom}({}_BU_A, {}_BB)]_B$. *Let* $\{u_i, i=1,\cdots,n\}$, $\{v_i, i=1,\cdots,n\}$ *be dual sets of generators of* ${}_BU$ *and* V_B *with respect to* ω_B. *Set*

$$S(X)={}_BU_A\otimes X,\ S'(X)={}_BU'_A\otimes X,\ T(Y)={}_AV_B\otimes Y. \tag{60}$$

Then the transfer homomorphism t *defined by the maps* λ *and* μ *below:*

$$[\lambda(g)](x)=\sum_{i=1}^{n}v_i\otimes g(u_i\otimes x), \qquad x\in X,$$

$$[\mu(f)](y)=\sum_{j=1}^{m}u'_j\otimes f(v'_j\otimes y), \qquad y\in Y,$$

where $\{v'_j\}$ *and* $\{u'_j\}$ *are dual sets of generators of* ${}_AV$ *and* U'_A *with respect to* ω_A, *is expressed as follows:*

$$[t(g)](x)=\sum_{i=1}^{n}\alpha_1(X')(v_i\otimes g(u_i\otimes x)), \tag{61}$$

$$\alpha_1(X')(v\otimes u'\otimes x)=\omega_A(v, u')x. \tag{62}$$

This lemma is readily obtained from Theorem 4.1 and Corollary 3.3.

In case A is an (S_0, T_0)-quasi-Frobenius extension of B, we can set

$${}_AV_B={}_AA_B,\ {}_BU'_A={}_BA_A,\ \omega_A(v, u')=vu',$$

and hence we have

$$[t(g)](x)=\sum_{i=1}^{n}v_ig(u_i\otimes x), \tag{63}$$

where ${}_BA_A\otimes X$ is identified with X.

Thus the transfer homomorphism coincides with Spur homomorphism in the sense of Kasch [7][11].

11) In the case of group rings, the transfer homomorphism in our sense coincides with that given by Cartan-Eilenberg [3, p. 254], and with the norm homomorphism as defined in [3, p. 233].

With Lemmas 6.7 and 6.8, Theorem 6.5 generalizes a theorem of Gaschütz-Ikeda-Kasch [6] as well as a theorem of B. Pareigis [16, Satz 11]; the results of B. Müller [13] are contained also in Theorem 6.5 (cf. the proof of Lemma 6.7).

EXAMPLE 6.9. In Lemma 6.7, (a)′ cannot be replaced by "X is S-projective". This is seen from the following example.

Let A be a subalgebra of the full matrix ring $(K)_8$ over a commutative field K which is generated by

$$m=c_{21}+c_{43}+c_{75}+c_{86},\qquad n=c_{31}+c_{42}+c_{65}+c_{87},$$

$$p=c_{41}+c_{85},\qquad q=c_{51}+c_{62}+c_{73}+c_{84},$$

together with 1, where c_{ik} are matrix units. Then $\{1, m, n, p, q, mq, nq, pq\}$ is a K-basis of A. Let B and C be subalgebras of A defined as $B=K1+Km$, $C=B+Bn$, and let α be an automorphism of A defined as $\alpha(m)=n$, $\alpha(n)=m$, $\alpha(p)=p$, $\alpha(q)=q$. Then A is an $(\alpha, 1)$-Frobenius extension of C and C is a $(1,1)$-Frobenius extension of B. Hence by Corollary 9.2 in §9 below A is an $(\alpha, 1)$-Frobenius extension of B. Let us set $N_0=Km$ and $L={}_AA_B\otimes{}_BN_0$. Then L is a left A-module such that $S_0(L)$ is S-projective, where S_0 is defined by (46), and $S(X)\cong{}_BA_A\otimes S_0^{-1}(X)$. Since A is a completely primary quasi-Frobenius algebra, L is indecomposable. If L were S-projective, then $S_0^{-1}(L)$ would be SS_0-projective. Since B is a completely primary uni-serial algebra and $S_0^{-1}(L)$ is indecomposable, $S_0^{-1}(L)$ would be A-isomorphic either to ${}_AA$ or to L. On the other hand, $S_0^{-1}(L)\cong S_0(L)\cong\alpha(L)$ and $\alpha(L)$ is A-isomorphic neither to ${}_AA$ nor to L since $(mq)[\alpha(L)]=0$ while $(mq)m\neq 0$, $m\in L$, $m\in{}_AA$. Thus L is not S-projective.

§7. Duality

Throughout this section, let S, S', T, ${}_BU_A$, ${}_AV_B$, ω, $\{u_1,\cdots,u_n\}$, $\{v_1,\cdots,v_n\}$ be the same as in Theorem 3.2.

Then by setting $Y={}_BB$ in Theorem 3.2 we see that

(64) $$\lambda:\ [\mathrm{Hom}_B({}_BU_A\otimes X, {}_BB_B)]_B\to[\mathrm{Hom}_A(X, {}_AV_B)]_B$$

defined by

(65) $$\lambda(g)(x)=\sum_{i=1}^n v_ig(u_i\otimes x)$$

is a right B-isomorphism which is natural in $X\in{}_A\mathfrak{M}$; this is seen from the naturality of λ in Theorem 3.2 with respect to X and Y. Here ${}_AV_B\otimes{}_BB$ is identified with ${}_AV_B$.

From (65) it follows that the inverse of λ is given by

(66) $$[\lambda^{-1}(f)](u\otimes x)=\omega(u, f(x))$$

for $x\in X$, $u\in U$, $f\in\mathrm{Hom}_A(X, {}_AV_B)$.

Similarly, the map λ' defined by

(65)′ $$\lambda'(g)(x)=\sum_{i=1}^{n}g(x\otimes v_i)u_i, \qquad \text{for } x\in X$$

is a left B-isomorphism

(64)′ $$\lambda':\ {}_B[\mathrm{Hom}_B(X\otimes_A V_B, B_B)]\cong{}_B[\mathrm{Hom}_A(X, {}_BU_A)]$$

which is natural in $X\in\mathfrak{M}_A$, and its inverse is given by

(66)′ $$[(\lambda')^{-1}(f)](x\otimes v)=\omega(f(x), v)$$

where $x\in X$, $v\in V$, $f\in\mathrm{Hom}_A(X, {}_BU_A)$.

LEMMA 7.1. *If ${}_AW$ is finitely generated and projective, then the map*

$$\rho:\ \mathrm{Hom}_A({}_AX, {}_AA)\otimes_A W\to\mathrm{Hom}_A({}_AX, {}_AW)$$

defined by

$$[\rho(f\otimes w)](x)=f(x)w$$

for $x\in X$, $w\in W$, $f\in\mathrm{Hom}_A({}_AX, {}_AA)$, is an isomorphism which is natural in $X\in{}_A\mathfrak{M}$ and in ${}_AW$.

PROOF. The naturality of ρ in X and in W is obvious. In case $W={}_AA$, ρ is clearly an isomorphism. Hence ρ is an isomorphism if ${}_AW$ is finitely generated and projective.

Now, let us assume that ${}_AV$ and U_A are finitely generated and projective. Then the map ρ defined by

(67) $$[\rho(f\otimes v)](x)=f(x)v$$

yield a right B-isomorphism

(68) $$\rho:\ [\mathrm{Hom}_A(X, {}_AA)\otimes_A V_B]_B\cong[\mathrm{Hom}_A(X, {}_AV_B)]_B$$

which is natural in $X\in{}_A\mathfrak{M}$.

Similarly, the map ρ' defined by

(67)′ $$[\rho'(u\otimes f)](x)=uf(x)$$

yields a left B-isomorphism

(68)′ $$\rho':\ {}_B[{}_BU_A\otimes\mathrm{Hom}_A(X, A_A)]\cong{}_B[\mathrm{Hom}_A(X, {}_BU_A)],$$

which is natural in $X\in\mathfrak{M}_A$.

Now, let us set

(69) $$\Phi=\mathrm{Hom}(\lambda, 1)\circ\mathrm{Hom}(\rho^{-1}, 1)\circ(\lambda')^{-1}\circ\rho',$$

where ρ' is a map defined by (68)′ with X replaced by $\mathrm{Hom}_A(X, {}_AA)$ ($X\in{}_A\mathfrak{M}$). Then

(70) $$\Phi:\ {}_BU_A\otimes\mathrm{Hom}_A([\mathrm{Hom}_A(X, {}_AA)]_A, A_A)$$
$$\to\mathrm{Hom}_B([\mathrm{Hom}_B({}_BU_A\otimes X, {}_BB)]_B, B_B)$$

is a left B-isomorphism which is natural in $X \in {}_A\mathfrak{M}$. Let

$$u \in U,\ v \in V,\ \alpha \in \mathrm{Hom}_A(\mathrm{Hom}_A(X, {}_AA), A_A),\ f \in \mathrm{Hom}_A(X, {}_AA).$$

Then by (65)′ and (67)′ we have

$$[(\lambda')^{-1}(\rho'(u \otimes \alpha))](f \otimes v) = \omega(\rho'(u \otimes \alpha)(f), v)$$

$$= \omega(u(\alpha(f)), v) = \omega(u, \alpha(f)v).$$

Now, let $g \in \mathrm{Hom}_B({}_BU_A \otimes X, {}_BB)$. If

(71) $$\rho^{-1}(\lambda(g)) = \sum_{j=1}^{m} f_j \otimes \bar{v}_j,$$

where $f_j \in \mathrm{Hom}_A(X, {}_AA)$, $\bar{v}_j \in V$, then we have

$$\Phi(u \otimes \alpha)(g) = \sum_{j=1}^{m} [(\lambda')^{-1}(\rho'(u \otimes \alpha))](f_j \otimes \bar{v}_j)$$

$$= \sum_{j=1}^{m} \omega(u, \alpha(f_j)\bar{v}_j).$$

Let x_0 be an arbitrary element of X and set

$$\alpha_0(f) = f(x_0) \qquad \text{for } f \in \mathrm{Hom}_A(X, {}_AA).$$

Then we have clearly $\alpha_0 \in \mathrm{Hom}_A(\mathrm{Hom}_A(X, {}_AA), A_A)$. On the other hand, by (71) and (67) we have

$$\lambda(g)(x_0) = \sum_{j=1}^{m} f_j(x_0)\bar{v}_j = \sum_{j=1}^{m} \alpha_0(f_j)\bar{v}_j.$$

Therefore, we get finally by (66)

(72) $$\Phi(u \otimes \alpha_0)(g) = g(u \otimes x_0).$$

Now, let us define

(73) $$\pi(X): X \to \mathrm{Hom}_A(\mathrm{Hom}_A(X, {}_AA), A_A), \qquad X \in {}_A\mathfrak{M},$$

(74) $$\pi(Y): Y \to \mathrm{Hom}_B(\mathrm{Hom}_B(Y, {}_BB), B_B), \qquad Y \in {}_B\mathfrak{M},$$

by

$$[\pi(X)(x)](f) = f(x),\ [\pi(Y)(y)](g) = g(y)$$

where $x \in X$, $f \in \mathrm{Hom}_A(X, {}_AA)$, $y \in Y$, $g \in \mathrm{Hom}_B(Y, {}_BB)$. Then Lemma 7.2 below is a direct consequence of (72).

LEMMA 7.2. *Assume that U_A and ${}_AV$ are finitely generated and projective. Then we have*

$$\pi({}_BU_A \otimes X) = \Phi \circ (1 \otimes \pi(X))$$

for $X \in {}_A\mathfrak{M}$, *and* Φ *is a B-isomorphism which is natural in* X.

We are now in a position to establish the following theorem.

THEOREM 7.3. *Let* $\{S, T\}$ *be an adjoint pair such that* S *has a left adjoint,* S' *has a right adjoint and* S *is faithful*[12]. *Then* $\pi(X)$ *is an A-isomorphism if and only if* $\pi(S(X))$ *is a B-isomorphism, where* $X \in {}_A\mathfrak{M}$.

PROOF. By assumption U_A, ${}_AV$ are finitely generated and projective, and U_A is a generator. Let us set $D = \mathrm{End}_A(U_A)$. Then by the convention made in § 1 U is viewed as a D-A-bimodule. Then the functor P: ${}_A\mathfrak{M} \to {}_D\mathfrak{M}$ defined by $P(X) = {}_DU_A \otimes X$ is a category-isomorphism by Theorem 1.2. If $\pi({}_BU_A \otimes X)$ is a B-isomorphism, then $1 \otimes \pi(X)$ is a B-isomorphism by Lemma 7.2. This isomorphism may be considered as a left D-isomorphism:

$$ {}_DU_A \otimes X \cong {}_DU_A \otimes \mathrm{Hom}_A(\mathrm{Hom}_A(X, {}_AA), A_A), $$

since $(1 \otimes \pi(X))(u \otimes x) = u \otimes \pi(X)(x)$. As is shown above, P is a category-isomorphism, and hence $\pi(X)$ is an A-isomorphism. Since the "only if" part is obvious by Lemma 7.2, the proof of Theorem 7.3 is completed hereby.

THEOREM 7.4. *Under the same assumptions as in Theorem* 7.3, $\pi(X)$ *is an A-monomorphism if and only if* $\pi(S(X))$ *is a B-monomorphism where* $X \in {}_A\mathfrak{M}$.

This is readily seen from the proof of Theorem 7.3.

COROLLARY 7.5. *Under the same assumptions as in Theorem* 7.3, *for a left ideal* J *of* A *we have* $l(r(J)) = J$ *if and only if we have* $l_\omega(r_\omega(UJ)) = UJ$. *Here* $r(L)$ *or* $l(L)$ *means the right or left annihilator of* L *in* A, *and*

$$ r_\omega(U') = \{v \in V \mid \omega(u', v) = 0 \quad \text{for all } u' \in U'\}, $$

$$ l_\omega(V') = \{u \in U \mid \omega(u, v') = 0 \quad \text{for all } v' \in V'\}. $$

PROOF. $\pi({}_A(A/J))$ is an A-monomorphism if and only if $l(r(J)) = J$, and $\pi({}_B(U/UJ))$ is a B-monomorphism if and only if $l_\omega(r_\omega(UJ)) = UJ$. This proves Corollary 7.5 in view of Theorem 7.4.

In applications of Theorems 7.3 and 7.4 it is to be noted that $\pi(Y)$ ($Y \in {}_B\mathfrak{M}$) is a B-isomorphism either

(a) if Y is finitely generated and projective, or

(b) if B is quasi-Frobenius and Y is finitely generated[13].

§ 8. Nakayama isomorphism

Let ${}_BU_A$, ${}_AV_B$, ω, $\{u_1, \cdots, u_n\}$, $\{v_1, \cdots, v_n\}$ be the same as described in Theorem 3.2. Let us set

$$ C = \mathrm{End}_B(V_B),\ D = \mathrm{End}_A(U_A),\ E = [\mathrm{End}_A({}_AV)]^0. \tag{75} $$

Then by the convention made in § 1 U is a D-A-bimodule and V is a C-B-bimodule

12) As is easily seen from the existence of an algebra which is QF-3 but not quasi-Frobenius, Theorem 7.3 does not hold in general unless S is faithful.

13) Cf. K. Morita and H. Tachikawa [12], Morita [10].

as well as an A-E-bimodule. Then by Lemma 2.3 C is ring-isomorphic to $[\mathrm{End}_B({}_BU)]^0$ which will be identified with C by the formula

(76) $$\omega(uc, v)=\omega(u, cv), \qquad \text{for } u\in U,\ v\in V.$$

Furthermore, we assume that

(77) $${}_BU,\ V_B \text{ and } {}_AV \text{ are faithful.}$$

Thus we may, and shall, consider that A is a subring of C and B is a subring of D and E.

Let us set

(78) $$C_0=Z_C(A),\ D_0=Z_D(B),\ E_0=Z_E(B),$$

where $Z_R(F)$ means the centralizer of F in R for a subset F of a ring R: $Z_R(F)=\{r\in R \mid rx=xr \text{ for all } x\in F\}$.

Let $d\in D_0$. Then the correspondence $u\to du$ defines a B-endomorphism of U and hence there is an element $\varphi(d)$ of C such that

(79) $$du=u\varphi(d) \qquad \text{for } u\in U.$$

It is clear that $\varphi(d)\in C_0$. Moreover, as is easily seen, φ is an inverse-isomorphism of the ring D_0 onto C_0. Similarly, there is an inverse-isomorphism ψ of the ring E_0 onto C_0 such that

(80) $$ve=\psi(e)v \qquad \text{for } v\in V,\ e\in E_0.$$

Now, let us set

(81) $$\theta(d)=(\psi^{-1}\circ\varphi)(d), \qquad d\in D_0.$$

Then θ is a ring-isomorphism of D_0 onto E_0 and we have

(82) $$\omega(du, v)=\omega(u, v\theta(d)), \qquad \text{for } u\in U,\ v\in V.$$

Since for $d\in D_0\cap B$ it holds that $\omega(du, v)=d\omega(u, v)=\omega(u, v)d=\omega(u, vd)$, we have

(83) $$\theta(d)=d, \qquad \text{for } d\in D_0\cap B.$$

Let us define a two-sided B-homomorphism h_0: ${}_BU_A\otimes{}_AV_B\to{}_BB_B$ by

(84) $$h_0(u\otimes v)=\omega(u, v) \qquad \text{for } u\in U,\ v\in V.$$

Let $h\in\mathrm{Hom}_{(B,B)}({}_BU_A\otimes{}_AV_B, {}_BB_B)$; then there is an element d of D_0 such that

$$h(u\otimes v)=\omega(du, v) \qquad \text{for } u\in U,\ v\in V,$$

since the correspondence $v\to h(u\otimes v)$ determines an element u' of U such that $h(u\otimes v)=\omega(u', v)$ for all v, and the correspondence $u\to u'$ is a B-A-endomorphism of U. Thus we have

(85) $$\mathrm{Hom}_{(B,B)}({}_BU_A\otimes{}_AV_B, {}_BB_B)=h_0D_0.$$

Let τ': ${}_AV_B \cong {}_A[\mathrm{Hom}_B({}_BU_A, {}_BB)]_B$ be another A-B-isomorphism and let $\omega'(u, v) = [\tau'(v)](u)$ for $u \in U$, $v \in V$. Then, as is seen from the above consideration, there is an element d_0 of D_0 such that d_0^{-1} exists and $\omega'(u, v) = \omega(d_0 u, v)$. Hence the ring-isomorphism θ' of D_0 onto E_0 corresponding to ω' is expressed as

$$\theta'(d) = \theta(d_0 d d_0^{-1}) \qquad \text{for } d \in D_0.$$

Thus θ is determined uniquely up to an inner automorphism. We shall call θ a *Nakayama isomorphism*.

Next, we shall consider

$$(64) \qquad \lambda\colon \mathrm{Hom}_B({}_BU_A \otimes X, {}_BB) \cong \mathrm{Hom}_A(X, {}_AV_B)$$

which is defined by (65) in § 7.

THEOREM 8.1. *The isomorphism λ in* (64) *is a right B-isomorphism which is natural in $X \in {}_A\mathfrak{M}$. If we regard the left-hand side of* (64) *as a right D_0-module and the right-hand side as a right E_0-module, then λ is a semi-linear isomorphism with respect to θ; that is,*

$$(86) \qquad [\lambda(gd)](x) = [\lambda(g)(x)]\theta(d), \qquad \textit{for } d \in D_0,\ x \in X.$$

Furthermore, λ maps $\mathrm{Hom}_{(B,B)}({}_BU_A \otimes {}_AX_B, {}_BB_B)$ *isomorphically onto* $\mathrm{Hom}_{(A,B)}({}_AX_B, {}_AV_B)$.

PROOF. From the property of dual sets of generators we get for $d \in D_0$

$$(87) \qquad \begin{cases} du_i = \sum_{j=1}^{n} \omega(du_i, v_j) u_j, \\ v_j \theta(d) = \sum_{i=1}^{n} v_i \omega(u_i, v_j \theta(d)). \end{cases}$$

Hence by virtue of (82) we have (86). The last part of the theorem follows readily from (65).

We note further that

$$(88) \qquad \lambda(h_0) = 1_V,$$

since $\lambda(h_0)(v) = \sum v_i h_0(u_i \otimes v) = \sum v_i \omega(u_i, v) = v$.

THEOREM 8.2. *Let ${}_AX_B$ be an A-B-bimodule which is A-B-isomorphic to a direct summand of a direct sum of a finite number of copies of ${}_AV_B$. Then the map*

$$\Phi\colon \mathrm{Hom}_{(B,B)}({}_BU_A \otimes {}_AX_B, {}_BB_B) \otimes_{D_0} [\theta, \mathrm{Hom}_A({}_AV_E, X')] \to \mathrm{Hom}_A(X, X')$$

defined by

$$[\Phi(f \otimes g)](x) = g(\lambda(f)(x)), \qquad x \in X$$

is an isomorphism which is natural in ${}_AX_B$ and in $X' \in {}_A\mathfrak{M}$.

Here for a left E-module Y we denote by (θ, Y) the left D_0-module which coincides with Y as an additive group and on which left multiplication by an element d of D_0 is defined by $d * y = \theta(d) y$, $y \in Y$.

PROOF. Let $h \in \mathrm{Hom}_{(A,B)}({}_AX_B, {}_AX''_B)$.

Then we have

$$\lambda(f''\circ(1\otimes h))=\lambda(f'')\circ h$$

for $f''\in \mathrm{Hom}_{(B,B)}({}_BU_A\otimes_A X''_B, {}_BB_B)$, where $1\otimes h\in \mathrm{Hom}_{(B,B)}({}_BU_A\otimes_A X_B, {}_BU_A\otimes_A X''_B)$. Thus Φ is natural in ${}_AX_B$. It is obvious that Φ is natural in $X'\in {}_A\mathfrak{M}$. Now, let us set ${}_AX_B={}_AV_B$.
Then we have for $g\in[\theta, \mathrm{Hom}_A({}_AV_E, X')]$

$$[\Phi(h_0d\otimes g)](v)=g(\lambda(h_0d)(v))=g((\lambda(h_0)(v))\theta(d))=g(v\theta(d))=(dg)(v),$$

that is, $\Phi(h_0d\otimes g)=dg$. This shows that Φ is an isomorphism in this case. Since Φ is natural in ${}_AX_B$, it follows that Φ is an isomorphism in general for ${}_AX_B$ with the type described in the theorem.

LEMMA 8.3. *Let X be a left A-module which is A-isomorphic to a direct summand of a finite direct sum of copies of ${}_AV$. Then the map*

$$\Psi:\ \mathrm{Hom}_A(X, {}_AV_E)\otimes_E[\mathrm{Hom}_A({}_AV_E, X')]\to \mathrm{Hom}_A(X, X')$$

defined by

$$[\Psi(f\otimes g)](x)=g(f(x)),\qquad x\in X,$$

is an isomorphism which is natural in X and $X'\in {}_A\mathfrak{M}$.

PROOF. If $X={}_AV$, then we have for $e\in E$

$$[\Phi(1_Ve\otimes g)](x)=g(xe)=(eg)(x),\qquad x\in X,$$

and hence Ψ is an isomorphism. Since Ψ is natural in X, the lemma follows readily.

On the basis of these results we can establish a cohomology theory for adjoint pairs, analogously as in F. Kasch [8] and B. Müller [13]. The case of QF-3 algebras treated by H. Tachikawa [17] is also contained in our considerations.

§ 9. The endomorphism ring theorem

Let

$$S, S':\ {}_A\mathfrak{M}\to{}_B\mathfrak{M},\qquad T:\ {}_B\mathfrak{M}\to{}_A\mathfrak{M},$$

$$P, P':\ {}_B\mathfrak{M}\to{}_C\mathfrak{M},\qquad Q:\ {}_C\mathfrak{M}\to{}_B\mathfrak{M}$$

be functors.

LEMMA 9.1. *Let $\{S, T\}$ and $\{P, Q\}$ be adjoint pairs such that S' is a right adjoint of T and P' is a right adjoint of Q. Then $\{PS, TQ\}$ is an adjoint pair such that $P'S'$ is a right adjoint of TQ.*

PROOF. We have

$$\mathrm{Hom}_C(PS(X), Z)\cong\mathrm{Hom}_B(S(X), Q(Z))\cong\mathrm{Hom}_A(X, TQ(Z)),$$

and similarly $\mathrm{Hom}_A(TQ(Z), X)\cong\mathrm{Hom}_C(Z, P'S'(X))$, where $X\in{}_A\mathfrak{M}$, $Y\in{}_B\mathfrak{M}$, $Z\in{}_C\mathfrak{M}$.

COROLLARY 9.2. *Under the same assumption as in Lemma* 9.1, *if* $\{S, T\}$ *is an* (S_0, T_0)-*quasi-strongly* (*resp.* (S_0, T_0)-*strongly*) *adjoint pair and* $\{P, Q\}$ *is a* (T_0, Q_0)-*quasi-strongly* (*resp.* (T_0, Q_0)-*strongly*) *adjoint pair, then* $\{PS, TQ\}$ *is an* (S_0, Q_0)-*quasi-strongly* (*resp.* (S_0, Q_0)-*strongly*) *adjoint pair, where* Q_0 *is a category-isomorphism from* ${}_C\mathfrak{M}$ *to itself.*

Proof is obvious.

We shall now establish the endomorphism ring theorem.

THEOREM 9.3. *Let* $\{S, T\}$ *be an* (S_0, T_0)-*quasi-strongly* (*resp.* (S_0, T_0)-*strongly*) *adjoint pair; let* ${}_BU_A$, ${}_AV_B$, ${}_BU'_A$ *be bimodules satisfying conditions* (36) *to* (39) *and* (40) (*resp.* (40)$''$). *Let us set*

$$C=\mathrm{End}_B(V_B)$$

and assume that ${}_AV$ *is faithful. By the convention made in* §1 A *becomes a subring of* C. *Then, if either* V_B *is a generator or* $T_0=1$, *then* C *is a* (Q_0, S_0)-*quasi-Frobenius* (*resp.* (Q_0, S_0)-*Frobenius*) *extension of* B, *where* Q_0 *is a category-isomorphism from* ${}_C\mathfrak{M}$ *to itself and* $Q_0=1$ *in case* $T_0=1$.

PROOF. By (38) we may assume that C is a right operator domain of U and $\mathrm{Hom}_B({}_CV_B, B_B)\cong{}_BU_C$. Then we have

$${}_CV_B\otimes{}_BU_C\cong{}_C[\mathrm{Hom}_B({}_BU_C, {}_BU_C)]_C\cong{}_CC_C.$$

Let us set

$$P(Y)=\mathrm{Hom}_B({}_BU_C, Y)\cong{}_CV_B\otimes Y, \qquad Y\in{}_B\mathfrak{M},$$

$$Q(Z)={}_BU_C\otimes Z, \qquad Z\in{}_C\mathfrak{M}.$$

Then we have

(91) $$PS(X)\cong{}_CC_A\otimes X,\ \ TQ(Z)\cong{}_AC_C\otimes Z,$$

(92) $$\mathrm{Hom}_A(TQ(Z), X)\cong\mathrm{Hom}_C(Z, PS'(X)).$$

Let us set

$$Q_0=PT_0Q \qquad \text{or } Q_0=1$$

according as ${}_BU$ is a generator (case i)) or $T_0=1$ (case ii)). In case i), P and Q are category-isomorphisms and $PT_0^{-1}SS_0$ is naturally equivalent to $Q_0^{-1}(PS)S_0$. Since S' is similar to $T_0^{-1}SS_0$ by assumption, PS' is similar to $Q_0^{-1}(PS)S_0$. Hence by (92) there exists a left adjoint R of PS and R is similar to $S_0(TQ)Q_0^{-1}$. In case ii), PS' is similar to $(PS)S_0$ and hence PS has a left adjoint which is similar to $S_0(TQ)$. In view of (92), we see by Theorem 5.1 that in either case C is a (Q_0, S_0)-quasi-Frobenius extension of A, and that if $\{S, T\}$ is an (S_0, T)-strongly adjoint pair, then C is a (Q_0, S_0)-Frobenius extension.

THEOREM 9.4. *Let* ${}_AV_B$ *be an* A-B-*bimodule such that*
1) V_B *is a finitely generated, projective generator and* 2) ${}_AV$ *is faithful.*

Let $C=\mathrm{End}_B(V_B)$. *If* C *is a* (Q_0, S_0)-*quasi-Frobenius* (*resp.* (Q_0, S_0)-*Frobenius*) *ex-*

tension of A, then $\{S, T\}$ is an (S_0, T_0)-quasi-strongly (resp. (S_0, T_0)-strongly) adjoint pair, where

$$S(X)=\mathrm{Hom}_B({}_AV_B, B_B)\otimes X,\ T(Y)={}_AV_B\otimes Y, \qquad X\in{}_A\mathfrak{M},\ Y\in{}_B\mathfrak{M}$$

and $T_0=QQ_0P$, $P(Y)={}_CV_B\otimes Y$, $Q(Z)=\mathrm{Hom}_C({}_CV_B, Z)$ $(Y\in{}_B\mathfrak{M}, Z\in{}_C\mathfrak{M})$, and Q_0: ${}_C\mathfrak{M}\to{}_C\mathfrak{M}$ is a category-isomorphism.

PROOF. Let us set

$$K(X)={}_CC_A\otimes X,\ L(Z)={}_AC_C\otimes Z, \qquad X\in{}_A\mathfrak{M},\ Z\in{}_C\mathfrak{M},$$

and let K': ${}_C\mathfrak{M}\to{}_A\mathfrak{M}$ be a right adjoint of L. Then by assumption $\{K, L\}$ is an (S_0, Q_0)-quasi-strongly (resp. (S_0, Q_0)-strongly) adjoint pair. On the other hand, P and Q are category-isomorphisms which are inverses of each other, and hence $\{Q, P\}$ is a strongly adjoint pair, and consequently $\{Q, P\}$ is a (Q_0, T_0)-strongly adjoint pair. Since $LP(Y)\cong T(Y)$, $\{S, T\}$ is an (S_0, T_0)-quasi-strongly (resp. (S_0, T_0)-strongly) adjoint pair by Corollary 9.2.

In view of Theorem 5.1, Theorems 9.3 and 9.4 contain Kasch's theorem [6, Satz 5] and its generalizations by Kasch [7], Nakayama-Tsuzuku [14], Pareigis [16] and Müller [13][14]. If A is a $(1, \beta)$-Frobenius extension of B and β is extendable to an automorphism α of A, A is an $(\alpha^{-1}, 1)$-Frobenius extension of B (cf. the end of §5), and hence $C=\mathrm{End}_B(A_B)$ is a $(1, \alpha^{-1})$-Frobenius extension of B by Theorem 9.3; this is the case treated by Nakayama-Tsuzuku [14] and Pareigis [16]. For an (α, β)-Frobenius extension A of B, $C=\mathrm{End}_B(A_B)$ is a (γ, α)-Frobenius extension of A in case A_B is free; except for this case and the case $\beta=1$, C is not always a (γ, α)-Frobenius extension of B where γ is an automorphism of C.

As an immediate consequence of Theorem 9.3 we have

COROLLARY 9.5. *Let $\{S, T\}$ and C be the same as in Theorem 9.3. If V_B is a generator and ${}_AV$ is faithful, then the functor T is the composite of a category-isomorphism from ${}_B\mathfrak{M}$ to ${}_C\mathfrak{M}$ and a functor ${}_AC_C\otimes Z$: ${}_C\mathfrak{M}\to{}_A\mathfrak{M}$ associated with a (Q_0, S_0)-quasi-Frobenius (resp. (Q_0, S_0)-Frobenius) extension C of A.*

Finally, in case B is a subring of A and there is an (S_0, T_0)-strongly adjoint pair such that $T(Y)\cong{}_AV_B\otimes Y$ and ${}_AV_B$ is similar to ${}_AA_B$, we shall say that A is a *strongly (S_0, T_0)-quasi-Frobenius extension* of B. Then we have

THEOREM 9.6. *If A is a strongly (S_0, T_0)-quasi-Frobenius extension of B, then $D=[\mathrm{End}_A({}_AV)]^0$ is a (Q_0', T_0)-Frobenius extension of B and the ring A is similar to D*[15].

PROOF. $\{T, S'\}$ is a (T_0, S_0)-strongly adjoint pair and the theorem follows from Theorem 9.3.

The notion of strongly (S_0, T_0)-quasi-Frobenius extensions is narrower than that of (S_0, T_0)-quasi-Frobenius extensions, but has much more similarity to quasi-Frobenius algebras as Theorem 9.6 shows.

14) Strictly speaking, Müller's result concerning two-sided quasi-Frobenius extensions. We can establish the theorems so that they may contain Müller's result concerning one-sided quasi-Frobenius extensions. Cf. also footnote 10).

15) Two rings E and F are called *similar* if there is a category-isomorphism from ${}_E\mathfrak{M}$ to ${}_F\mathfrak{M}$.

§ 10. Adjoint pairs of functors and category-isomorphisms

We shall first prove the following theorem.

THEOREM 10.1. *Let* $\{S, T\}$ *be an adjoint pair such that* S *has a left adjoint,* S' *has a right adjoint* T', *and* S' *is faithful. Let* $Y \in {}_B\mathfrak{M}$.

(a) *If* Y *is projective or injective, so is* $T(Y)$.

(b) *If* Y *is finitely generated, faithful, or a generator, so is* $T(Y)$.

(c) *In case* T' *is faithful, if* $\mathrm{l.dim}\, Y = m < \infty$, *then* $\mathrm{l.dim}\, T(Y) = m$.

PROOF. By Theorem 4.2 there exist bimodules ${}_AV_B$, ${}_BU_A$, ${}_BU'_A$ which satisfy conditions (37) to (39) and we have

$$T(Y) \cong {}_AV_B \otimes Y, \quad S'(X) \cong \mathrm{Hom}_A({}_AV_B, X) \cong {}_BU'_A \otimes X.$$

Furthermore, ${}_AV$ is a generator by Lemma 1.5. Hence U'_A is also a generator. Thus (b) follows readily from Lemmas 2.4 and 1.3. Next, (a) is a direct consequence of Lemmas 6.6 and 6.6′. Finally, to prove (c) we first observe that there is a natural isomorphism

$$\mathrm{Ext}^n_A(T(Y), X) \cong \mathrm{Ext}^n_B(Y, S'(X)), \qquad n = 0, 1, 2, \cdots. \tag{93}$$

Secondly, since $T'(Y) \cong \mathrm{Hom}({}_BU'_A, Y)$, $S'(X) \cong {}_BU'_A \otimes X$ and T' is faithful, ${}_BU'$ is a generator and hence for any $Y_0 \in {}_B\mathfrak{M}$ there is $X_0 \in {}_A\mathfrak{M}$ such that $S'(X_0)$ is B-homomorphically mapped onto Y_0. The first fact implies $\mathrm{l.dim}\, T(Y) \leqq m$, while the second implies $\mathrm{l.dim}\, T(Y) \geqq m$.

THEOREM 10.2. *Under the same assumptions as in Theorem* 10.1, *the following statements are true.*

(a) *If* B *satisfies the minimum condition for left* (*resp. right*) *ideals, so does* A.

(b) *If* B *is a quasi-Frobenius ring, so is* A.

(c) *In case* S *is faithful, if* B *is an* S*-ring in the sense of Kasch* [6], *so is* A.

PROOF. Let ${}_AV_B$ and ${}_BU'_A$ be the bimodules as described in the proof of Theorem 10.1. Then with the notations of Theorem 4.3 we have

$$\tilde{S}(X) = X \otimes {}_AV_B \cong \mathrm{Hom}_A({}_BU'_A, X), \qquad X \in \mathfrak{M}_A.$$

Since U'_A is a generator, $\tilde{S}: \mathfrak{M}_A \to \mathfrak{M}_B$ is faithful by Lemma 1.5. Now, (a) follows from the fact that S' (resp. $\tilde{S}$) is a faithful exact functor. Next, (b) is a direct consequence of Theorem 10.1.

Finally, to prove (c) we first observe that a ring C with the minimum condition for left and right ideals is an S-ring if and only if for any non-zero left (resp. right) module Z we have $\mathrm{Hom}_C(Z, {}_CC) \neq 0$ (resp. $\mathrm{Hom}_C(Z, C_C) \neq 0$). If B is an S-ring, and X is a non-zero left A-module, then $S(X) \neq 0$ and we have by Lemma 7.1

$$0 \neq \mathrm{Hom}_B(S(X), {}_BB) \cong \mathrm{Hom}_A(X, {}_AV_B) \cong \mathrm{Hom}_A(X, {}_AA) \otimes {}_AV_B.$$

Hence $\mathrm{Hom}_A(X, {}_AA) \neq 0$. Similarly, we can prove, by making use of $\tilde{S}$, that $\mathrm{Hom}_A(X, A_A) \neq 0$ for a non-zero right A-module X. Thus A is an S-ring.

Theorem 10.1 shows that a number of properties for modules are preserved under a functor which is a member of an adjoint pair with some conditions. From this point of view we can give a characterization of category-isomorphisms.

THEOREM 10.3. *Let $\{S, T\}$ be an adjoint pair of functors such that S' has a right adjoint. Then T is a category-isomorphism if and only if the conditions* a) *and* b) *below are satisfied*:

a) *if $X \in {}_A\mathfrak{M}$ is simple, so are $S(X)$ and $S'(X)$*;

b) *if $Y \in {}_B\mathfrak{M}$ is simple, so is $T(Y)$.*

Here A and B are assumed to satisfy the minimum condition for left ideals.

PROOF. We have only to prove the "if" part. Suppose that a) and b) hold. Let $N(A)$ (resp. $N(B)$) be the radical of A (resp. B) and set

$$\bar{A}\bar{e}_i = Ae_i/N(A)e_i, \quad \bar{B}\bar{f}_j = Bf_j/N(B)f_j,$$

where $\{e_1, \cdots, e_m\}$ (resp. $\{f_1, \cdots, f_n\}$) is a maximal set of mutually orthogonal primitive idempotents of A (resp. B) such that $Ae_i \not\cong Ae_j$ (resp. $Bf_i \not\cong Bf_j$) for $i \neq j$. By assumption there is a map π from $\{1, \cdots, m\}$ into $\{1, \cdots, n\}$ such that $S(\bar{A}\bar{e}_i) \cong \bar{B}\bar{f}_{\pi(i)}$, $i = 1, \cdots, m$. Since $\mathrm{Hom}_B(S(X), Y) \cong \mathrm{Hom}_A(X, T(Y))$, we have $\mathrm{Hom}_A(\bar{A}\bar{e}_i, T(\bar{B}\bar{f}_{\pi(i)})) \neq 0$ and hence $\bar{A}\bar{e}_i \cong T(\bar{B}\bar{f}_{\pi(i)})$. Hence π is one-to-one and onto. Thus we may, and shall, assume that $m = n$ and

$$S(\bar{A}\bar{e}_i) \cong \bar{B}\bar{f}_i, \quad T(\bar{B}\bar{f}_i) \cong \bar{A}\bar{e}_i, \qquad i = 1, \cdots, m.$$

From this it follows further that $S'(\bar{A}\bar{e}_i) \cong \bar{B}\bar{f}_i$. Since we have

$$\begin{aligned} \mathrm{Hom}_B(S(Ae_i), \bar{B}\bar{f}_j) &\cong \mathrm{Hom}_A(Ae_i, T(\bar{B}\bar{f}_j)) \\ &\cong \mathrm{Hom}_A(Ae_i, \bar{A}\bar{e}_j) \cong \begin{cases} \bar{e}_i\bar{A}\bar{e}_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \end{aligned}$$

and $S(Ae_i)$ is projective by Lemma 6.6, we see that $S(Ae_i) \cong Bf_i$. Similarly, $T(Bf_i) \cong Ae_i$, $i = 1, \cdots, m$, since S' is exact.

Now let us consider the map $\alpha_0(X)$ defined by (49) in §6. It is seen that $\alpha_0(\bar{A}\bar{e}_i)\colon \bar{A}\bar{e}_i \to TS(\bar{A}\bar{e}_i)$ is an A-isomorphism since it is not zero as the image of the identity map of $S(\bar{A}\bar{e}_i)$ onto itself under λ. Let ρ be the canonical projection from Ae_i onto $\bar{A}\bar{e}_i$. Then the diagram

$$\begin{tikzcd} 0 \arrow[r] & N(A)e_i \arrow[r] \arrow[d, "\alpha_0(N(A)e_i)"] & Ae_i \arrow[r, "\rho"] \arrow[d, "\alpha_0(Ae_i)"] & \bar{A}\bar{e}_i \arrow[r] \arrow[d, "\alpha_0(\bar{A}\bar{e}_i)"] & 0 \\ & TS(N(A)e_i)) \arrow[r] & TS(Ae_i) \arrow[r, "TS(\rho)"] & TS(\bar{A}\bar{e}_i) \arrow[r] & 0 \end{tikzcd}$$

is commututive and each row is exact. Since $TS(\rho) \circ \alpha_0(Ae_i) = \alpha_0(\bar{A}\bar{e}_i) \circ \rho \neq 0$, $\alpha_0(Ae_i)$ is an epimorphism; otherwise we would have $\mathrm{Image}\, \alpha_0(Ae_i) \subseteq \mathrm{Kernel}\, TS(\rho)$, since $TS(\bar{A}\bar{e}_i) \cong \bar{A}\bar{e}_i$, $TS(Ae_i) \cong Ae_i$ as has been proved above. Hence $\alpha_0(Ae_i)$ is an A-isomorphism. Consequently, $\alpha_0({}_AA)$ is an A-isomorphism. Similarly, $\beta_1({}_BB)$ is a B-isomorphism.

In view of Theorem 4.2 we may assume that

$$S(X)={}_BU_A\otimes X,\ T(Y)={}_AV_B\otimes Y,\ S'(X)={}_BU'_A\otimes X$$

where ${}_BU_A$, ${}_BU'_A$ and ${}_AV_B$ are bimodules with properties described in Theorem 4.2. Then from the naturality of $\alpha_0(X)$ it follows that $\alpha_0({}_AA)$ is a two-sided A-isomorphism: $\alpha_0({}_AA):{}_AA_A\cong{}_AV_B\otimes{}_BU_A$. This shows that TS is naturally equivalent to the identity functor. Similarly, $\beta_1({}_BB)$ is a two-sided B-isomorphism and ST is naturally equivalent to the identity functor. Therefore T is a category-isomorphism. This proves Theorem 10.3[16].

In case there are functors $S^{(n)}: {}_A\mathfrak{M}\to{}_B\mathfrak{M}$ and $T^{(n)}: {}_B\mathfrak{M}\to{}_A\mathfrak{M}$, $n=0, \pm1, \pm2,\cdots$, such that $S^{(0)}=S$, $T^{(0)}=T$ and $S^{(n)}$ is a left adjoint of $T^{(n)}$ and a right adjoint of $T^{(n-1)}$, and each of $S^{(n)}$ and $T^{(n)}$ are faithful, we shall call each of S and T a *category-quasi-isomorphism*. As is easily seen, if $\{S, T\}$ is an (S_0, T_0)-quasi-strongly adjoint pair and if S and T are faithful then S and T are category-quasi-isomorphism. In case there is a category-quasi-isomorphism from ${}_A\mathfrak{M}$ to ${}_B\mathfrak{M}$, we shall say that A is *quasi-similar* to B; quasi-similarity is an equivalence relation. Theorem 10.3 may be considered as a characterization of category-isomorphisms among category-quasi-isomorphisms. Theorem 10.1 as well as Theorem 7.3 gives a number of properties of modules preserved under category-quasi-isomorphisms, while the properties of rings mentioned in Theorem 10.2 and the finistic left global dimension of a ring (cf. [1]) are quasi-similarity invariant.

References

[1] H. BASS, Finistic dimension and a homological characterization of semi-primary rings. Trans. Amer. Math Soc., **95** (1960), 466–488.

[2] ———, The Morita theorems. Lecture notes, Summer Institute on Algebra 1962, University of Oregon, Eugene, Oregon, U. S. A.

[3] H. CARTAN and S. EILENBERG, Homological algebra. Princeton, 1956.

[4] C. W. CURTIS and I. REINER, Representation theory of finite groups and associative algebras. New York, 1962.

[5] D. M. KAN, Adjoint functors. Trans. Amer. Math. Soc., **87** (1958), 294–323.

[6] F. KASCH, Grundlagen einer Theorie der Frobenius-Erweiterungen. Math. Ann., **127** (1954), 453–474.

[7] ———, Projektive Frobenius-Erweiterungen. Sitzungsber. Heidelberger Akad. Wiss., (1960/1961), 89–109.

[8] ———, Dualitätseigenschaften von Frobenius-Erweiterungen. Math. Z., **77** (1961), 219–227.

[9] S. MACLANE, Homology. Springer, 1963.

[10] K. MORITA, Duality for modules and its applications to the theory of rings with minimum condition. Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, **6**, No. 150 (1958), 83–142.

16) Even in case $\{S, T\}$ is a strongly adjoint pair, Theorem 10.3 fails to hold if condition a) is dropped. For example, let B be a commutative field and let A be a division algebra over B and set $S(X)={}_BA_A\otimes X$, $T(Y)={}_AA_B\otimes Y$; then all the assumptions of Theorem 10.3 are satisfied except a) but T is not a category-isomorphism if $A\neq B$.

[11] K. MORITA, Category-isomorphisms and endomorphism rings of modules. Trans. Amer. Math. Soc., **103** (1962), 451–469.
[12] ——— and H. TACHIKAWA, Character modules, submodules of a free module, and quasi-Frobenius rings. Math. Z., **65** (1956), 414–428.
[13] B. MÜLLER, Quasi-Frobenius-Erweiterungen. Math. Z., **85** (1964), 345–368.
[14] T. NAKAYAMA and T. TSUZUKU, On Frobenius extensions. I, II. Nagoya Math. J., **17** (1960), 89–110; **19** (1961), 127–148.
[15] ——— and ———, Corrections to our paper "On Frobenius extensions. II". Nagoya Math. J., **20** (1962), 205.
[16] B. PAREIGIS, Einige Bemerkungen über Frobenius-Erweiterungen. Math. Ann., **153** (1964), 1–13.
[17] H. TACHIKAWA, On dominant dimensions of QF-3 algebras. Trans. Amer. Math. Soc., **112** (1964), 249–266.