Products of normal spaces with metric spaces. II

To Professor Y. Akizuki on his Sixtieth Birthday

By

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(Received June 20, 1963)

The topological product of a normal space with a metrizable space is not normal in general, as has been shown recently by E. Michael [1]. In a previous paper [4][1] we have introduced the notion of P-spaces, and established that a necessary and sufficient condition for a normal space X to possess the property that the product space $X\times Y$ is normal for any metrizable space Y is that X be a P-space.

In this paper we shall discuss basic coverings in the sense defined below for the topological product of a normal space with a metric space, and a necessary and sufficient condition for the topological product of a normal space with a metric space to be countably paracompact and normal (resp. paracompact and normal) will be established in terms of basic coverings. As an application we shall give a characterization of metric spaces whose product with any countably paracompact normal space is normal.

§1. Basic coverings

Let Y be a metrizable space. Let $\mathfrak{V}=\cup\mathfrak{V}_i$ be an open basis of Y such that i) $\mathfrak{V}_i=\{V_{i\alpha}\mid\alpha\in\Omega_i\}$ is a σ-locally finite open covering of Y for $i=1, 2, \cdots$, and ii) $\{\mathrm{St}(y, \mathfrak{V}_i)\mid i=1, 2, \cdots\}$ is a basis for neighborhoods at each point y of Y; the existence of such an open basis $\mathfrak{V}=\cup\mathfrak{V}_i$, for which each $\mathfrak{V}_i$ is locally finite, is assured by a well-knwon theorem of A. H. Stone. Let us put

$$W(\alpha_1, \cdots, \alpha_i)=\bigcap_{\nu=1}^{i} V_{\nu\alpha_\nu}, \qquad \text{for} \qquad \alpha_1\in\Omega_1, \cdots, \alpha_i\in\Omega_i. \tag{1}$$

Let X be a normal space. We shall say that a covering $\mathfrak{G}$ of the product space $X\times Y$ is a basic covering if $\mathfrak{G}$ has the form

$$\mathfrak{G}=\{G(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)\mid\alpha_\nu\in\Omega_\nu, \nu=1, \cdots, i;\ i=1, 2, \cdots\} \tag{2}$$

and if $\{G(\alpha_1, \cdots, \alpha_i)\mid\alpha_\nu\in\Omega_\nu,\ \nu=1, \cdots, i;\ i=1, 2, \cdots\}$ is a family of open subsets of X such that

$$G(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}) \quad \text{for} \quad \alpha_1\in\Omega_1, \cdots, \alpha_{i+1}\in\Omega_{i+1}. \tag{3}$$

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1) The contents of [4] were announced in [3].

In case for a basic covering $\mathfrak{G}$ in (2) there exists a family $\{F(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, \nu=1, \cdots, i;\ i=1, 2, \cdots\}$ of closed subsets of X such that

$$(4) \qquad F(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i)$$

$$(5) \qquad \cup\{F(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu,\ \nu=1, \cdots, i;\ i=1, 2, \cdots\} = X\times Y,$$

we shall say that $\mathfrak{G}$ has a special refinement.

LEMMA 1.1. *For a basic covering $\mathfrak{G}$ in (2) the following three statements are equivalent.*

(a) *$\mathfrak{G}$ has a special refinement.*

(b) *There exists a family $\{F(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu,\ \nu=1, \cdots, i;\ i=1, 2, \cdots\}$ of open F_σ-subsets of X satisfying* (4) *and* (5).

(c) *There exists a family $\{F(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu,\ \nu=1, \cdots, i;\ i=1, 2, \cdots\}$ of F_σ-subsets of X satisfying* (4) *and* (5).

PROOF. The implications (a)→(b) and (b)→(c) are obvious. Assume (c). Then there exist closed subsets $C(\alpha_1, \cdots, \alpha_i; k)$ of X such that

$$F(\alpha_1, \cdots, \alpha_i) = \bigcup_{k=1}^{\infty} C(\alpha_1, \cdots, \alpha_i;\ k)$$

If we put

$$K(\alpha_1, \cdots, \alpha_i) = \cup\{C(\alpha_1, \cdots, \alpha_j;\ k) \mid j \leqq i,\ k \leqq i\}$$

then $K(\alpha_1, \cdots, \alpha_i)$ are closed subsets of X, and we have clearly $K(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i)$ in view of (3) and (4). Let (x, y) be any point of $X\times Y$. Then we have $(x, y) \in F(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)$ for some $\alpha_1 \in \Omega_1, \cdots, \alpha_i \in \Omega_i$. If $x \in C(\alpha_1, \cdots, \alpha_i;\ k)$ for some k, we put Max $(i, k)=j$, and select elements $\alpha_{i+1} \in \Omega_{i+1}, \cdots, \alpha_j \in \Omega_j$ so that $y \in V_{\nu\alpha_\nu}$, $\nu=i+1, \cdots, j$. Then we have

$$(x, y) \in K(\alpha_1, \cdots, \alpha_j) \times W(\alpha_1, \cdots, \alpha_j).$$

Thus $\{K(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu,\ \nu=1, \cdots, i;\ i=1, 2, \cdots\}$ is a covering of $X\times Y$, and hence (a) holds.

Now we shall prove the following theorem which is fundamental in this paper.

THEOREM 1.2. *Let X be a normal space and Y a metrizable space. Then a basic covering of $X\times Y$ has a special refinement if and only if it is a normal covering.*

PROOF. Since a basic covering is a σ-locally finite open covering, the "only if" part is a direct consequence of Lemma 1.1 and Morita [4, Theorem 1.2]. To prove the "if" part, suppose that $\mathfrak{G}$ in (2) is a normal covering of $X\times Y$.

Then there exists a locally finite open covering

$$(6) \qquad \mathfrak{L} = \{L_\lambda \mid \lambda \in \Lambda\}$$

of $X\times Y$ such that

$$(7) \qquad \{\bar{L}_\lambda \mid \lambda \in \Lambda\} \text{ is a refinement of } \mathfrak{G}.$$

Now, let us denote by $L(\alpha_1, \cdots, \alpha_i;\ \lambda)$ the union of all the open subsets P of X such that $P\times W(\alpha_1, \cdots, \alpha_i) \subset L_\lambda$: that is,

(8) $$L(\alpha_1, \cdots, \alpha_i;\ \lambda)=\cup\{P \mid P\times W(\alpha_1, \cdots, \alpha_i)\subset L_\lambda,\ P \text{ open in } X\},$$

for the case where $W(\alpha_1, \cdots, \alpha_i)\neq 0$. In case $W(\alpha_1, \cdots, \alpha_i)=0$, we put

(9) $$L(\alpha_1, \cdots, \alpha_i;\ \lambda) = 0\,.$$

Then we have

(10) $$L(\alpha_1, \cdots, \alpha_i;\ \lambda) \times W(\alpha_1, \cdots, \alpha_i)\subset L_\lambda\,,$$

(11) $$\cup\{L(\alpha_1, \cdots, \alpha_i;\ \lambda) \times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu\in\Omega_\nu,\ \nu=1, \cdots, i;\ i = 1, 2, \cdots;\ \lambda\in\Lambda\}=X\times Y.$$

Let us put

(12) $$F(\alpha_1, \cdots, \alpha_i)=0, \text{ in case } W(\alpha_1, \cdots, \alpha_i)=0\,,$$

(13) $$F(\alpha_1, \cdots, \alpha_i)=\cup\{\overline{L(\alpha_1, \cdots, \alpha_i;\ \lambda)} \mid \overline{L(\alpha_1, \cdots, \alpha_i;\ \lambda)}\subset G(\alpha_1, \cdots, \alpha_i)\,,\ \lambda\in\Lambda\}, \text{ in case } W(\alpha_1, \cdots, \alpha_i)\neq 0\,.$$

Since $L(\alpha_1, \cdots, \alpha_i;\ \lambda)\times W(\alpha_1, \cdots, \alpha_i)\subset L_\lambda$ and $\{L_\lambda\}$ is locally finite in $X\times Y$, the set $F(\alpha_1, \cdots, \alpha_i)$ defined by (13) is closed in case $W(\alpha_1, \cdots, \alpha_i)\neq 0$. Thus $F(\alpha_1, \cdots, \alpha_i)$ is closed in each case, and we have by (12) and (13)

(14) $$F(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i)\,.$$

Let (x, y) be any point of $X\times Y$. Then there exists some $\lambda\in\Lambda$ such that $(x, y)\in L_\lambda$. From (7) we have

(15) $$\bar{L}_\lambda\subset G(\alpha_1, \cdots, \alpha_i) \times W(\alpha_1, \cdots, \alpha_i)$$

for some $\alpha_1\in\Omega_1, \cdots, \alpha_i\in\Omega_i$. Since $(x, y)\in L_\lambda$ and $y\in W(\alpha_1, \cdots, \alpha_i)$, there exist $\alpha_{i+1}\in\Omega_{i+1}, \cdots, \alpha_j\in\Omega_j$ and an open neighborhood $U(x)$ of x such that

(16) $$(x, y)\in U(x) \times W(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j)\subset L_\lambda\,.$$

Then from the definition of $L(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j;\ \lambda)$ it follows that $U(x)\subset L(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j;\ \lambda)$. Hence we have

(17) $$(x, y)\in L(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j;\ \lambda)\times W(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j)\subset L_\lambda\,.$$

Therefore by (15) we see that

(18) $$\overline{L(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j;\ \lambda)}\subset G(\alpha_1, \cdots, \alpha_i)\,.$$

Consequently we have in view of (3)

(19) $$\overline{L(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j;\ \lambda)}\subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j)\,,$$

and hence

(20) $$\overline{L(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j;\ \lambda)}\subset F(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j)\,.$$

From (17) and (20) it follows that

(21) $$(x, y)\in F(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j)\times W(\alpha_1, \cdots, \alpha_i, \alpha_{i+1}, \cdots, \alpha_j)\,.$$

Since (x, y) is an arbitrary point of $X\times Y$, we have

$$(22)\qquad \cup\{F(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu\in\Omega_\nu,\ \nu=1, \cdots, i;\ i=1, 2, \cdots\}=X\times Y.$$

Thus we see, in view of (14), (22), that $\mathfrak{G}$ in (2) has a special refinement, and the "if" part of Theorem 1. 2 is proved.

§2. The product of a normal space with a metric space

Now we shall prove the following theorem.

THEOREM 2. 1. *Let X be a normal space and Y a metrizable space. If the product space $X\times Y$ is countably paracompact and normal, then every basic covering of $X\times Y$ has a special refinement.*

PROOF. Every basic covering of $X\times Y$ is a σ-locally finite open covering of $X\times Y$, and every σ-locally finite open covering of a countably paracompact normal space is a normal covering by [4, Lemma 1. 5]. Hence Theorem 2. 1 follows directly from Theorem 1. 2.

THEOREM 2. 2. *Let X be a normal space and Y a metrizabie space. Suppose that every basic covering of $X\times Y$ has a special refinement. Then the product space $X\times Y$ is normal. Furthermore, if X is $\mathfrak{m}$-paracompact, then $X\times Y$ is also $\mathfrak{m}$-paracompact.*[2)]

PROOF.[3)] Let X be $\mathfrak{m}$-paracompact and normal ($\mathfrak{m}\geqq 2$), and let $\mathfrak{M}=\{M_\lambda \mid \lambda\in\Lambda\}$ ($2\leqq|\Lambda|\leqq\mathfrak{m}$) be any open covering of $X\times Y$. Let us denote by $M(\alpha_1, \cdots, \alpha_i;\ \lambda)$ the union of all the open subsets P of X such that $P\times W(\alpha_1, \cdots, \alpha_i)\subset M_\lambda$. We put

$$(23)\qquad M(\alpha_1, \cdots, \alpha_i)=\cup\{M(\alpha_1, \cdots, \alpha_i;\ \lambda) \mid \lambda\in\Lambda\}.$$

Then

$$(24)\qquad \{M(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu\in\Omega_\nu,\ \nu=1, \cdots, i;\ i=1, 2, \cdots\}$$

is a basic covering of $X\times Y$. Hence from the assumption of the theorem and Lemma 1. 1 it follows that there exists a family $\{H(\alpha_1, \cdots, \alpha_i)\}$ of open F_σ-subsets of X such that

$$(25)\qquad H(\alpha_1, \cdots, \alpha_i)\subset M(\alpha_1, \cdots, \alpha_i)$$

and

$$(26)\qquad \{H(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu\in\Omega_\nu,\ \nu=1, \cdots, i;\ i=1, 2, \cdots\}$$

is a covering of $X\times Y$. By [4, Theorem 1. 2] the covering (26) is normal. Each subspace $H(\alpha_1, \cdots, \alpha_i)$ is $\mathfrak{m}$-paracompact and normal by [4, Theorem 1. 3]. Hence $\{M(\alpha_1, \cdots, \alpha_i;\ \lambda)\frown H(\alpha_1, \cdots, \alpha_i) \mid \lambda\in\Lambda\}$ is a normal covering of $H(\alpha_1, \cdots, \alpha_i)$. Therefore

$$(27)\qquad \{[M(\alpha_1, \cdots, \alpha_i;\ \lambda)\frown H(\alpha_1, \cdots, \alpha_i)]\times W(\alpha_1, \cdots, \alpha_i) \mid \lambda\in\Lambda\}$$

is a normal covering of $H(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)$ by [4, Lemma 1. 4]. Since $M(\alpha_1, \cdots, \alpha_i;\ \lambda)\times W(\alpha_1, \cdots, \alpha_i)\subset M_\lambda$, we see by [4, Theorem 1. 1] that $\{M_\lambda\}$ is a normal

2) A topological space is called $\mathfrak{m}$-paracompact if every open covering consisting of at most $\mathfrak{m}$ sets has a locally finite open covering as a refinement.

3) This proof is similar to our proof of [4, Lemma 4. 4].

covering of $X\times Y$. This completes the proof of Theorem 2. 2.

Combining Theorems 2. 1 and 2. 2 we obtain the following theorem.

THEOREM 2. 3. *Let X be a topological space and Y a metrizable space. In order that the product space $X\times Y$ be $\mathfrak{m}$-paracompact and normal (resp. paracompact and normal) it is necessary and sufficient that X be $\mathfrak{m}$-paracompact and normal (resp. paracompact and normal) and every basic covering of $X\times Y$ have a special refinement. Here $\mathfrak{m}\geqq\aleph_0$.*

As an immediate consequence of Theorem 2. 3 we have

THEOREM 2. 4. *Let X be a topological space and Y a metrizable space. Then the product space $X\times Y$ is $\mathfrak{m}$-paracompact and normal (resp. paracompact and normal) if and only if X is $\mathfrak{m}$-paracompact and normal (resp. paracompact and normal) and $X\times Y$ is countably paracompact and normal. Here $\mathfrak{m}\geqq\aleph_0$.*

As for the Lindelöf property we obtain the theorems analogous to Theorems 2. 3 and 2. 4; Theorem 2. 5 may be proved simiarly as in [4, Theorem 5. 3].

THEOREM 2. 5. *Let X be a regular space and Y a separable metrizable space. In order that the product space $X\times Y$ have the Lindelöf property it is necessary and sufficient that X have the Lindelöf property and every basic covering of $X\times Y$ have a special refinement.*

THEOREM 2. 6. *The product of a topological space X and a separable metrizable space Y is a regular Lindelöf space if and only if X is regular Lindelöf and $X\times Y$ is countably paracompact and normal.*

§3. Metric spaces whose product with any countably paracompact normal space is normal

We shall first prove

LEMMA 3. 1. *Let X be a normal space and Y a metrizable space. Suppose that $\{B_i\mid i=1, 2, \cdots\}$ is any (not necessarily open or closed) countable covering of Y and that $X\times B_i$ is countably paracompact and normal for $i=1, 2, \cdots$. Then $X\times Y$ is also countably paracompact and normal.*

PROOF. With the same notations as in §1, let

$$(2)\qquad \mathfrak{G}=\{G(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)\mid \alpha_\nu\in\Omega_\nu,\ \nu=1, \cdots, i\,;\ i=1, 2, \cdots\}$$

be any basic covering of $X\times Y$. Then

$$(28)\qquad \{G(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)\frown B_j\mid \alpha_\nu\in\Omega_\nu,\ \nu=1, \cdots i\,;\ i=1, 2, \cdots\}$$

is a basic covering of $X\times B_j$. Hence by Theorem 2. 3 there exists a family

$$(29)\qquad \{F_j(\alpha_1, \cdots, \alpha_i)\mid \alpha_\nu\in\Omega_\nu,\ \nu=1, \cdots, i\,;\ i=1, 2, \cdots\}$$

of closed subsets of X such that

$$(30)\qquad F_j(\alpha_1, \cdots, \alpha_i)\subset G(\alpha_1, \cdots, \alpha_i)$$

$$(31)\qquad \smile\{F_j(\alpha_1, \cdots, \alpha_i)\times W(\alpha_1, \cdots, \alpha_i)\frown B_j\mid \alpha_\nu\in\Omega_\nu,\ \nu=1, \cdots, i\,;\ i=1, 2, \cdots\}=X\times B_j\,.$$

If we put

$$F(\alpha_1, \cdots, \alpha_i) = \cup\{F_j(\alpha_1, \cdots, \alpha_i) \mid j=1, 2, \cdots\},$$

then $F(\alpha_1, \cdots, \alpha_i)$ are F_σ-subsets of X and satisfy the conditions below:

$$F(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i) \tag{32}$$

$$\cup\{F(\alpha_1, \cdots, \alpha_i) \times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, \nu=1, \cdots, i;\ i=1, 2, \cdots\} = X \times Y. \tag{33}$$

Therefore $\mathfrak{G}$ has a special refinement by Lemma 1. 1, and our Lemma 3. 1 follows directly from Theorem 2. 3 (in the case $\mathfrak{m} = \aleph_0$).

Now we give a characterization of metric spaces whose product with any countably paracompact normal space is normal.

THEOREM 3. 2. *Let Y be a metrizable space. In order that the product space $X \times Y$ be normal for any countably paracompact normal space X, it is necessary and sufficient that Y be a countable union of locally compact subsets.*

PROOF. The necessity of the condition follows immediately from recent results of E. Michael [1] and A. H. Stone [5]. Since the product of a countably paracompact normal space with a locally compact metrizable space is countably paracompact and normal by [2, Theorem 5], the sufficiency of the condition of the theorem is a direct consequence of Lemma 3. 1.

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