On the Kernel Functions for Symmetric Domains

By

Kiiti MORITA

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The importance of the Bergman kernel functions has been recognized in the theory of functions of one and several complex variables ([1]). In the present paper, after proving some general theorems concerning kernel functions, we shall determine the kernel functions for the four main types of irreducible bounded symmetric domains[1]. According to E. Carton [2], any bounded symmetric domain is expressed as the topological product of irreducible domains, and hence our results, if the kernel function will be determined for the two exceptional cases, will yield a complete information about the kernel functions of bounded symmetric domains by virtue of Theorem 3 below, which asserts that the kernel function of the topological product of two domains is equal to the product of the kernel functions of two domains.

The classical Schwarz lemma asserts that if a function $f(z)$ in a complex variable z is regular in the domain $|z|<1$ and $|f(z)|<1$ in $|z|<1$, then the inequalities

$$\text{(a)} \qquad \rho(f(z_1), f(z_2)) \leqq \rho(z_1, z_2)$$

$$\text{(b)} \qquad |f'(z)| \leqq \frac{1-|f(z)|^2}{1-|z|^2}$$

hold, where ρ denotes the non-Euclidean distance in the interior of the unit circle.

In previous papers [6], [7] (cf. also M. Sugawara [11]) we have established the validity of (a) for any analytic mapping f of D into itself in case D is one of the matrix spaces included in the four main types of irreducible symmetric domains. In the present paper we shall therefore prove the validity of (a) for any analytic mapping f of D into itself in case D is a complex sphere. Our concern lies, of course, in the deduction of the theorem to the effect that if the equality sign in (a) holds for every point z_2 in some neighbourhood of one point, f is necessarily an analytical homeomorphism. This will be stated as Theorem 8.

In this connection the full group of all analytical homeomorphisms of a complex sphere $\mathfrak{M}_{(n)}$ onto itself will be determined. It is to be noted that the full group of analytical homeomorphisms was determined for every matrix domain previously ([6], [7], [9], [11]).

As for the relation (b) we shall give a generalization of it in terms of the Bergman kernel functions (Theorem 4).

1) The kernel functions for the matrix spaces were determined in a recent paper by J. Mitchell [4]. Our results were obtained in 1944 and quoted in Mitchell [4]; our method is different from Mitchell's.

Finally we shall give some remarks on the Laplacian corresponding to the Bergman metric. The Laplacian is shown to be invariant under any analytical homeomorphism of the domain onto itself, and a harmonic function, which may be considered as a generalization of Poisson's kernel, will be constructed in terms of the kernel functions for each of the four main types of irreducible bounded symmetric domains. The Cauchy formula due to S. Bochner [13] will be obtained from Poisson's integral formula by reversing J. Mitchell's argument [12].

1. Kernel functions. Let D be a bounded domain in a finite dimensional complex Euclidean space and let $\mathfrak{L}^2(D)$ be the class of all functions f which are regular in D and for which the Lebesgue integral $\int_D |f(z)|^2 dv_z = \|f\|^2 < \infty$. Here dv_z means the Euclidean volume element at z. Then $\mathfrak{L}^2(D)$ is a Hilbert space[2]. Since $\mathfrak{L}^2(D)$ is separable, there exists a complete orthonormal system $\{\varphi_n | n=1, 2, \cdots\}$. Then we define the kernel function of D after S. Bergman by

$$K_D(x, \bar{y}) = \sum_{\nu=1}^{\infty} \varphi_\nu(x)\overline{\varphi_\nu(y)} \; ; \tag{1}$$

the convergence is easily verified[3]. This function is independent of the choice of a complete orthonormal system $\{\varphi_n\}$. For a fixed point y in D, $K_D(x, \bar{y})$ as a function of x belongs to $\mathfrak{L}^2(D)$ and

$$(Pg)(x) = \int_D K(x, \bar{y}) g(y) dv_y \, , \qquad g \in L^2(D) \, ,$$

defines the projection operator of $L^2(D)$ upon $\mathfrak{L}^2(D)$, where $L^2(D)$ is the class of all Lebesgue measurable functions f on D for which $\int_D |f|^2 dv < \infty$. In particular we have

$$f(x) = \int_D K(x, \bar{y}) f(y) \, dv_y \, , \qquad f \in \mathfrak{L}^2(D) \, .$$

This is the so-called reproducing property.

Theorem 1. *For any function f belonging to $\mathfrak{L}^2(D)$ and for any positive number $\varepsilon > 0$ there exist a finite number of complex numbers $\alpha_1, \cdots, \alpha_s$ and points $y_1, \cdots, y_s$ in D such that*

$$\int_D |f(z) - \sum_{j=1}^{s} \alpha_j K_D(z, \bar{y}_j)|^2 dv_z < \varepsilon \, .^{2)}$$

Proof. Let us denote by $\mathfrak{F}$ the minimal closed linear manifold which contains $K_D(x, \bar{y})$ for every $y \in D$ (here $K_D(x, \bar{y})$ being considered as a function in x). Then we have $\mathfrak{F} = \mathfrak{L}^2(D)$. Because if $\mathfrak{F} \neq \mathfrak{L}^2(D)$ there would exist a function g such that

2) If, $\|f_n\| < C$, $n=1, 2, \cdots$, for a suitable constant C, then $\{f_n\}$ converges to f weakly in the Hilbert space $\mathfrak{L}^2(D)$ if and only if $\{f_n\}$ converges to f uniformly in every closed region in D.

3) Unless otherwise stated, x, y, X, Y are generally points or matrices with *complex* numbers as coordinates or coefficients.

$g \in \mathfrak{L}^2(D)$, $g \in \mathfrak{F}$, and $\int_D g\bar{f}\,dv=0$ for every $f \in \mathfrak{F}$; since $K_D(x, y) \in \mathfrak{F}$ we would have then

$$g(y)=\int_D g(x)\overline{K_D(x, \bar{y})}\,dv_x=0 ,$$

and hence $g(y)\equiv 0$. This proves the theorem.

Theorem 2. *If we put*

$$\begin{aligned} d(y_1, y_2) &= \left[\int_D |K_D(x, \bar{y}_1)-K_D(x, \bar{y}_2)|^2\,dv_x\right]^{1/2} \\ &= [K_D(y_1, \bar{y}_1)+K_D(y_2, \bar{y}_2)-K_D(y_1, \bar{y}_2)-K_D(y_2, \bar{y}_1)]^{1/2} , \end{aligned}$$

then d defines a metric of D which induces the topology of D given as a subspace of the complex Euclidean space. (cf. **15**)

Proof. If $K_D(x, \bar{y}_1)=K_D(x, \bar{y}_2)$ for every $x \in D$, we have $f(y_1)=f(y_2)$ for every $f \in \mathfrak{L}^2(D)$, since

$$f(y_j)=\int_D f(x)\overline{K_D(x, \bar{y}_j)}\,dv_x , \qquad j=1, 2.$$

Therefore we have $y_1=y_2$. It is evident that d satisfies the remaining axioms for metric.

If a sequence $\{y_n\}$ of points of D converges to a point y_0 of D, then we have clearly $d(y_n, y_0)\to 0$.

Let us suppose that for points y_j, $j=0, 1, 2, \cdots$, of D,

$$\int_D |K_D(x, \bar{y}_n)-K_D(x, \bar{y}_0)|^2 dv_x \to 0$$

as $n\to\infty$. Then we have

$$\int_D g(x)\overline{K_D(x, \bar{y}_n)}\,dv_x \to \int_D g(x)\overline{K_D(x, \bar{y}_0)}dv_x$$

for any $g \in \mathfrak{L}^2(D)$. Therefore

$$g(y_n) \to g(y_0)$$

for any $g \in \mathfrak{L}^2(D)$.

Let z_0 be a limit point of a subsequence $\{y_{k_n}\}$ of $\{y_n\}$. Then z_0 belongs to the closure $\bar{D}$ of D. If we consider an open bounded domain G containing $\bar{D}$, we have

$$F(y_{k_n}) \to F(z_0)$$

for every $F \in \mathfrak{L}^2(G)$. Since $\mathfrak{L}^2(D) \supset \mathfrak{L}^2(G)$, we have

$$F(z_0)=F(y_0)$$

for every $F \in \mathfrak{L}^2(G)$. Therefore $y_0=z_0$. This shows that the sequence $\{y_n\}$ converges to y_0.

Thus Theorem 2 is completely proved.

Let D be the topological product of two bounded domains D_1 and D_2. Let

$\{\varphi_n\}$ and $\{\psi_m\}$ be complete orthonormal systems in $\mathfrak{L}^2(D_1)$ and $\mathfrak{L}^2(D_2)$ respectively. Then $\varphi_n(x)\psi_m(y)$ clearly belongs to $\mathfrak{L}^2(D_1\times D_2)$ for every pair (n, m). For any $f(x, y)\in\mathfrak{L}^2(D_1\times D_2)$ we put

$$a_{nm}=\iint_{D_1\times D_2} f(x, y)\,\overline{\varphi_n(x)}\,\overline{\psi_m(y)}\,dv_x dv_y ,$$

$$f_m(x)=\int_{D_2} f(x, y)\,\overline{\psi_m(y)}\,dv_y .$$

If we fix a point x we have

$$\sum_{m=1}^{\infty}|f_m(x)|^2=\int_{D_2}|f(x, y)|^2 dv_y ,$$

since $f(x, y)$ belongs to $\mathfrak{L}^2(D_2)$ as a regular function in y, and $\{\psi_m(y)\}$ is a complete orthonormal system. On the other hand, since $f(x, y)\in\mathfrak{L}^2(D_1\times D_2)$, we have, by a theorem of Fubini,

$$\int_{D_1}\sum_{m=1}^{\infty}|f_m(x)|^2 dv_x=\iint_{D_1\times D_2}|f(x, y)|^2 dv_x dv_y .$$

By a theorem of Lebesgue this is written as follows:

$$\sum_{m=1}^{\infty}\int_{D_1}|f_m(x)|^2 dv_x=\iint_{D_1\times D_2}|f(x, y)|^2 dv_x dv_y<+\infty .$$

Therefore $f_m(x)$ belongs to $\mathfrak{L}^2(D_1)$ for each m, and we have

$$\int_{D_1}|f_m(x)|^2 dv_x=\sum_{n=1}^{\infty}|a_{nm}|^2 ,$$

since

$$a_{nm}=\int_{D_1} f_m(x)\overline{\varphi_n(x)}\,dv_x .$$

Hence we have

$$\iint_{D_1\times D_2}|f(x, y)|^2 dv_x dv_y=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|a_{nm}|^2 .$$

This shows that $\{\varphi_n(x)\psi_m(y)\}$ is a complete orthonormal system in $\mathfrak{L}^2(D_1\times D_2)$.

Thus we obtain the following theorem.

Theorem 3. *If D is the topological product of two bounded domains D_1 and D_2, then the Hilbert space $\mathfrak{L}^2(D_1\times D_2)$ is the direct product of the Hilbert spaces $\mathfrak{L}^2(D_1)$ and $\mathfrak{L}^2(D_2)$, and we have*

$$(2)\qquad K_D((x_1, x_2), \overline{(y_1, y_2)})=K_{D_1}(x_1, \bar{y}_1)K_{D_2}(x_2, \bar{y}_2) ,$$

where x_j, $y_j\in D_j$, $j=1, 2$.

2. A generalization of the relation (b)

Theorem 4. *Let D be a bounded analytically homogeneous domain in p-dimensional complex Euclideans space and let f be an analytic mapping of D into itself. Then we have*

$$(3)\qquad K_D(f(z), f(z))\left|\frac{\partial(f(z))}{\partial(z)}\right|^2 \leqq K_D(z, \bar{z})\,,$$

where $\dfrac{\partial(f(z))}{\partial(z)}$ *means the Jacobian of the transformation* f *with respect to complex variables* $z=(z_1, \cdots, z_n)$. *Moreover, if the equality sign in* (3) *holds for at least one point* z, *then the mapping* f *is necessarily one-to one and onto.*

In case f is a one-to-one analytic mapping of D onto itself, as is well known, we have further

$$(4)\qquad K_D(f(x), f(y))\left(\frac{\partial(f(x))}{\partial(x)}\right)\left(\frac{\overline{\partial(f(y))}}{\partial(y)}\right) = K_D(x, \bar{y})\,.$$

By this transformation law Theorem 4 may be proved as follows: Let $w_0=f(z_0)$. Let us denote by φ an analytical homeomorphism of D onto itself such that $\varphi(w_0)=z_0$. Then the composite $g=\varphi\circ f$ of two maps f and φ is a mapping which leaves z_0 invariant. Therefore, by a theorem of C. Carathéodory and H. Cartan, we have $\left|\left[\dfrac{\partial(g(z))}{\partial(z)}\right]_{z=z_0}\right| \leqq 1$, and the equality sign holds if and only if g is a homeomorphism of D onto itself. On the other hand,

$$\left[\frac{\partial(g(z))}{\partial(z)}\right]_{z=z_0} = \left[\frac{\partial(\varphi(w))}{\partial(w)}\right]_{w=w_0} \cdot \left[\frac{\partial(f(z))}{\partial(z)}\right]_{z=z_0}\,.$$

Hence we obtain Theorem 4 by the transformation law (4).

3. Let us assume that D is a circular domain in the sense of H. Cartan with the origin $(0, 0, \cdots, 0)$ as its centre and that D is analytically homogeneous. Then a theorem of H. Cartan shows that there exists a complete orthonormal system $\{\varphi_0, \varphi_1, \varphi_2, \cdots\}$ in the Hilbert space $\mathfrak{L}^2(D)$ such that φ_0 is a constant and $\varphi_j (j\geqq 1)$ are homogeneous polynomials of $z_1, \cdots, z_n$ with degree $\geqq 1$. Let $v(D)$ be the Euclidean volume of the domain D. Then we have $\varphi_0 = v(D)^{-1/2}$ and hence

$$K_D(0, \bar{z}) = K_D(z, 0) = v(D)^{-1}$$

where 0 means the origin $(0, 0, \cdots, 0)$.

Let T_y be a one-to-one analytic mapping of D onto itself which carries the point y into 0. Then we obtain from (4)

$$(5)\qquad K_D(x, \bar{y}) = v(D)^{-1}\frac{\partial(T_y(x))}{\partial(x)} \cdot \frac{\overline{\partial(T_y(y))}}{\partial(y)}$$

and $K_D(x, \bar{y}) \neq 0$. Thus the determination of the kernel functions is reduced to the calculation of the Jacobian of the transformation T_y.

Remark. From the homogeneity of φ_n defined above we have

$$K_D(rx, \bar{y}) = K_D(x, r\bar{y})$$

if $rx, ry \in D$ for a real number r. Since D is a domain of regularity, D is a complete circular domain. If for any point z of the boundary of D, $rx \in D$ for every r such that $0\leqq r<1$, then $K_D(x, \bar{y})$ can be defined for $x\in D$, $y \in \bar{D}$ by (1);

where $\bar{D}$ is the closure of D, and is continuous for $x\in D$, $y\in\bar{D}$. Therefore if F is a compact set contained in D there exists a positive constant C depending upon F such that

$$C^{-1}\leqq\left|\frac{\partial(\sigma(x))}{\partial(x)}\right|\Big/\left|\frac{\partial(\sigma(y))}{\partial(y)}\right|\leqq C$$

for any two points x, $y\in F$ and for any analytical homeomorphism σ of D onto itself, since if σ carries a point a into the origin we have

$$\frac{\partial(\sigma(x))}{\partial(x)}\Big/\frac{\partial(\sigma(y))}{\partial(y)}=K_D(x,\bar{a})/K_D(y,\bar{a})$$

by (4). This leads to the distorsion theorem mentiond in Hua [3], since a topological product of irreducible domains described in 4 below satisfies the above assumption.

4. The four main types of irreducible bounded symmetric domains are as follows (cf. [2]).

I. $\mathfrak{A}_{(n,m)}$: The set of all matrices Z of type (n, m) such that $E^{(m)}-\bar{Z}'Z$ is positive definite $(n\geqq m)$.

II. $\mathfrak{S}_{(n)}$: The set of all symmetric matrices Z of order n such that $E^{(n)}-\bar{Z}'Z$ is positive definite.

III. $\mathfrak{L}_{(n)}$: The set of all skew-symmetric matrices Z of order n such that $E^{(n)}-\bar{Z}'Z$ is positive definite.

IV. $\mathfrak{M}_{(n)}$: The set of all matrices Z of type $(n, 1)$ (i.e. n-dimensional vectors) such that

$$|Z'Z|<1\,,\qquad 1-2\bar{Z}'Z+|Z'Z|^2>0\,.$$

Here $E^{(r)}$ denotes the unit matrix of order r and we mean by X' and $\bar{X}$ the transposed and the conjugate matrix of X respectively.

If we denote by $\sim$ the analytical equivalence (the existence of an analytical homeomorphism), then we have

1) $\mathfrak{A}_{(1,1)}\sim\mathfrak{S}_{(1)}\sim\mathfrak{L}_{(2)}\sim\mathfrak{M}_{(1)}$ 2) $\mathfrak{M}_{(2)}\sim\mathfrak{M}_{(1)}\times\mathfrak{M}_{(1)}$
3) $\mathfrak{A}_{(3,1)}\sim\mathfrak{L}_{(3)}$ 4) $\mathfrak{S}_{(2)}\sim\mathfrak{M}_{(3)}$
5) $\mathfrak{A}_{(2,2)}\sim\mathfrak{M}_{(4)}$ 6) $\mathfrak{L}_{(4)}\sim\mathfrak{M}_{(6)}$
7) $\mathfrak{A}_{(m,n)}\sim\mathfrak{A}_{(n,m)}$

and there are no other relations than these as is easily seen. The relation 6) is overlooked in E. Cartan's paper [2] and this is shown by the following correspondence:

$$\mathfrak{M}_{(6)}\ni Z=\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_6\end{pmatrix}\leftrightarrow\begin{pmatrix} 0 & z_1+iz_2 & z_3+iz_4 & z_5+iz_6\\ -(z_1+iz_2) & 0 & z_5-iz_6 & -z_3+iz_4\\ -(z_3+iz_4) & -(z_5-iz_6) & 0 & z_1-iz_2\\ -(z_5+iz_6) & -(-z_3+iz_4) & -(z_1-iz_2) & 0\end{pmatrix}=\mathfrak{Z}\in\mathfrak{L}_{(4)}$$

5. The kernel functions of the irreducible domains.

Theorem 5. *The kernel functions for the four main types of irreducible bounded symmetric domains are as follows:*

(6) $K(X, \overline{Y})=v(\mathfrak{A}_{(n,m)})^{-1} \det. (E^{(m)}-\overline{Y}'X)^{-(n+m)}$, for $\mathfrak{A}_{(n,m)}$;

(7) $K(X, \overline{Y})=v(\mathfrak{S}_{(n)})^{-1} \det. (E^{(n)}-\overline{Y}'X)^{-(n+1)}$, for $\mathfrak{S}_{(n)}$;

(8) $K(X, \overline{Y})=v(\mathfrak{L}_{(n)})^{-1} \det. (E^{(n)}-\overline{Y}'X)^{-(n-1)}$, for $\mathfrak{L}_{(n)}$;

(9) $K(X, \overline{Y})=v(\mathfrak{M}_{(n)})^{-1}(1-2\overline{Y}'X+\overline{Y'Y}\cdot X'X)^{-n}$, for $\mathfrak{M}_{(n)}$.

Here X and Y are arbitrary points of each domain.

6. Proof of Theorem 5 (I). Let $A\in\mathfrak{A}_{(n,m)}$. Then the transformation T_A which carries A into the zero matrix is of the form

$$W=N_A^{-1}(Z-A)(E^{(m)}-\overline{A}'Z)^{-1}M_A\,, \tag{10}$$

where N_A and M_A are positive definite Hermitian matrices of order n and m such that $N^2{}_A=E^{(n)}-A\overline{A}'$, $M^2{}_A=E^{(m)}-\overline{A}'A$. Then $dW=N_A(E^{(n)}-Z\overline{A}')^{-1}dZ(E^{(m)}-\overline{A}'Z)^{-1}M_A$ and the Jacobian of T_A is calculated as follows:

$$\frac{\partial(T_A(Z))}{\partial(Z)}=\det(M_A^{-1}(E^{(m)}-\overline{A}'Z))^{-(n+m)}\,,$$

Thus we have (6) by (5).

For the case $\mathfrak{S}_{(n)}$ or $\mathfrak{L}_{(n)}$ we have $M_A=\overline{N}_A$ and the Jacobian of T_A is

$$\det(\overline{N}_A^{-1}(E^{(n)}-\overline{A}'Z))^{-(n+1)} \quad\text{or}\quad \det(\overline{N}_A^{-1}(E^{(n)}-\overline{A}'Z))^{-(n-1)}\,,$$

and we have (7) and (8) by (5). (cf. [5], [6], [7], [10]).

7. The complex spheres. For the case $\mathfrak{M}_{(n)}(n\geqq 4)$ we shall proceed similarly.

Theorem 6. *Let U_1, U_2, U_3, U_4 be respectively real matrices of type (n, n), $(n, 2)$, $(2, n)$, $(2, 2)$ such that*

$$\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4\end{pmatrix}'\begin{pmatrix} E^{(n)} & 0 \\ 0 & -E^{(2)}\end{pmatrix}\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4\end{pmatrix}=\begin{pmatrix} E^{(n)} & 0 \\ 0 & -E^{(2)}\end{pmatrix}, \tag{11a}$$

$$\det. U_4>0\,.\text{[4)]} \tag{11b}$$

Then the transformation defined by

$$W=\left\{U_1Z+U_2\begin{pmatrix}\frac{1}{2}(Z'Z+1)\\ \frac{i}{2}(Z'Z-1)\end{pmatrix}\right\}\left\{(1,\ i)\left\{U_3Z+U_4\begin{pmatrix}\frac{1}{2}(Z'Z+1)\\ \frac{i}{2}(Z'Z-1)\end{pmatrix}\right\}\right\}^{-1}. \tag{12}$$

is a one-to-one analytic mapping of $\mathfrak{M}_{(n)}$ onto itself and, conversely, any one-to-one analytic mapping of $\mathfrak{M}_{(n)}$ onto itself is of this form.

This theorem will be proved in **13**.

4) The condition (11b) is missing in Hua's paper [3].

For two points X and Y of $\mathfrak{M}_{(n)}$ we define

$$K_0(X, \overline{Y})=1-2\overline{Y}'X+\overline{Y}'\overline{Y}\cdot X'X .$$

Then we have

$$|(X-Y)'(X-Y)|<|K_0(X, \overline{Y})| \tag{13}$$

for any two points X and Y of $\mathfrak{M}_{(n)}$, and in particular,

$$K_0(X, \overline{Y})\neq 0 .$$

Let $A\in\mathfrak{M}_{(n)}$. Then $K_0(A, \overline{A})E^{(n)}+(A\overline{A}'-\overline{A}A')^2$ is a positive definite real symmetric matrix and hence there exists a uniquely determined matrix H_A of order n such that H_A is a positive definite real symmetric matrix and $H_A^2=K_0(A, \overline{A})E^{(n)}+(A\overline{A}'-\overline{A}A')^2$. We have

$$H_A\cdot A=A(1-\overline{A}'A) . \tag{14}$$

Now let us put

$$U_A=\begin{pmatrix} U_A^{(1)} & U_A^{(2)} \\ U_A^{(3)} & U_A^{(4)} \end{pmatrix} \tag{15}$$

$$=\frac{1}{K_0(A, \overline{A})^{1/2}}\left(\begin{array}{c:cc} H_A+A\overline{A}'+\overline{A}A' & -(A+\overline{A}) & -i(A-\overline{A}) \\ \hdashline -(A'+\overline{A}') & 1+\frac{1}{2}(A'A+\overline{A}'\overline{A}) & \frac{i}{2}(A'A-\overline{A}'\overline{A}) \\ -i(A'-\overline{A}') & \frac{i}{2}(A'A-\overline{A}'\overline{A}) & 1-\frac{1}{2}(A'A+\overline{A}'\overline{A}) \end{array}\right)$$

Then this matrix satisfies the conditions (11a) and (11b) as is verified by (14) and the definition of H_A. The transformation corresponding to this matrix is

$$W=\frac{1}{K_0(Z, \overline{A})}\{(H_A+A\overline{A}'+\overline{A}A')Z-\overline{A}Z'Z-A\} , \tag{16}$$

and it can also be written as follows:

$$W=\frac{1}{K_0(Z, \overline{A})}\{(H_A+A\overline{A}'-\overline{A}A')(Z-A)-\overline{A}(Z-A)'(Z-A)\} . \tag{16$'$}$$

This transformation carries A into the zero vector. Thus we may consider it as T_A in the notation of **3**.

8. To prove directly that the transformation (16) is a one-to-one analytic mapping of $\mathfrak{M}_{(n)}$ onto itself we may utilize the following equalities for the transformation (16):

$$W'W=\frac{1}{K_0(Z, \overline{A})}(Z-A)'(Z-A) , \tag{17}$$

$$K_0(W, \overline{W})\cdot|K_0(Z, \overline{A})|^2=K_0(Z, \overline{Z})K_0(A, \overline{A}) . \tag{18}$$

Indeed, if $Z \in \mathfrak{M}_{(n)}$, then we have $|W'W|<1$ by (13) and (17), and since $K_0(X, \overline{X})>0$ for any point X of $\mathfrak{M}_{(n)}$ it follows from (18) that $K_0(W, \overline{W})>0$ and consequently we have $W \in \mathfrak{M}_{(n)}$. If we consider the transformation T_{-A} which is obtained from (16) by replacing A by $-A$, then T_{-A} is shown to be the inverse of (16). Hence the transformation (16) is one-to-one and onto. At the same time we see that $\mathfrak{M}_{(n)}$ is analytically homogeneous.

9. Let V be a real orthogonal matrix of order n and let θ be a real number. Then the transformation defined by

$$W=e^{i\theta}VZ \tag{19}$$

is a one-to-one mapping of $\mathfrak{M}_{(n)}$ onto itself, which will be denoted by $T_{V,\theta}$. It holds that

$$T_{V,\theta}T_A=T_{A^*}T_{V,\theta}\,, \quad \text{for } A^*=T_{V,\theta}A\,. \tag{20}$$

We need the following lemma.

Lemma 1. *If A, B are two points of $\mathfrak{M}_{(n)}$, then there exists a transformation $T_{V,\theta}$ such that*

$$T_{V,\theta}A=(\alpha_1, \sqrt{-1}\alpha_2)0, \cdots, 0)'\,, \qquad T_{V,\theta}B=(b_1, b_2, b_3, b_4, 0, \cdots, 0)'\,,$$

where α_1 and α_2 are real numbers.

Proof. If we put $a_j=\lambda_j+\sqrt{-1}\mu_j$ with real numbers λ_j, μ_j, there exist two real orthogonal matrices T_1 and T_2 such that

$$T_1\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_1 \\ \vdots & \vdots \\ \vdots & \vdots \\ \lambda_n & \mu_n \end{pmatrix}T_2=\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \qquad \text{det. } T_2=1\,,$$

where α_1 and α_2 are real numbers. If we write

$$T_2=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

we have

$$e^{i\theta}T_1\begin{pmatrix} \lambda_1+i\mu_1 \\ \lambda_2+i\mu_2 \\ \vdots \\ \vdots \end{pmatrix}=\begin{pmatrix} \alpha_1 \\ i\alpha_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If we have

$$\begin{pmatrix} b_1' \\ \vdots \\ b_n' \end{pmatrix}=e^{i\theta}T_1\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

we apply the above process to $\begin{pmatrix} b_3' \\ \vdots \\ b_n' \end{pmatrix}$; then we can prove the lemma.

Thus for the proof of (13), (17), (18) it is sufficient to prove them for

$$A=(\alpha_1, \sqrt{-1}\alpha_2, 0, \cdots, 0)' \quad \text{and} \quad Z=B=(b_1, b_2, b_3, b_4, 0, \cdots, 0)' .$$

In this case the matrix U_A and the transformation (16)' are calculated as follows:

$$U_A=\frac{1}{K_0(A, A)^{1/2}}$$

$$\times\left(\begin{array}{cc|ccc|cc} 1+\alpha_1^2-\alpha_2^2 & 0 & & & & -2\alpha_1 & 0 \\ 0 & 1-\alpha_1^2+\alpha_2^2 & & 0 & & 0 & 2\alpha_2 \\ \hline & & K_0(A, \bar{A})^{1/2} & & & & \\ & 0 & & \ddots & & & 0 \\ & & & & K_0(A, \bar{A})^{1/2} & & \\ \hline -2\alpha_1 & 0 & & & & 1+\alpha_1^2-\alpha_2^2 & 0 \\ 0 & 2\alpha_2 & & 0 & & 0 & 1-\alpha_1^2+\alpha_2^2 \end{array}\right)$$

$$(21)\quad \begin{cases} w_1=\dfrac{1}{K_0(Z, A)}\{(1-\alpha_1^2-\alpha_2^2)(z_1-\alpha_1)-2i\alpha_1\alpha_2(z_2-i\alpha_2)-\alpha_1(Z-A)'(Z-A)\} \\ w_2=\dfrac{1}{K_0(Z, \bar{A})}\{2i\alpha_1\alpha_2(z_1-\alpha_1)+(1-\alpha_1^2-\alpha_2^2)(z_2-i\alpha_2)+i\alpha_2(Z-A)'(Z-A)\} \\ w_j=\dfrac{K_0(A, \bar{A})^{1/2}}{K_0(Z, \bar{A})}z_j\,, \qquad j=3, 4, \cdots, n\,, \end{cases}$$

where $A=(\alpha_1, \sqrt{-1}\alpha_2, 0, \cdots, 0)'$.

10. The image C of B by the transformation (21) is of the form $(c_1, c_2, c_3, c_4, 0, \cdots, 0)'$. Hence if we put

$$A_0=(a_1, a_2, 0, 0)'\,, \qquad B_0=(b_1,b_2,b_3,b_4)'\,, \qquad C_0=(c_1, c_2, c_3, c_4)'\,,$$

we have $C_0=T_{A_0}\cdot B_0$. Thus the general case is reduced to the case $n=4$, since $K_0(B, \bar{A})=K_0(B_0, \bar{A}_0)$, $C'C=C_0'C_0$, $(B-A)'(B-A)=(B_0-A_0)'(B_0-A_0)$, etc.

As is stated in **4**, $\mathfrak{M}_{(4)}$ is analytically equivalent to $\mathfrak{A}_{(2,2)}$. The transformation τ:

$$(22)\qquad \mathfrak{M}_{(4)}\ni Z=\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}\longrightarrow\begin{pmatrix} z_1+iz_2 & z_3+iz_4 \\ -z_3+iz_4 & z_1-iz_2 \end{pmatrix}=\mathfrak{Z}\in\mathfrak{A}_{(2,2)}$$

establishes this analytical equivalence.

By τ, the transformation (21) for $n=4$ is brought into the form

(23) $$\mathfrak{W}=\mathfrak{N}^{-1}(\mathfrak{Z}-\mathfrak{A})(E^{(2)}-\overline{\mathfrak{A}}'\mathfrak{Z})^{-1}\mathfrak{N}\ ,$$

where $\mathfrak{N}$ is a positive definite real symmetric matrix such that $\mathfrak{N}^2=E^{(2)}-\overline{\mathfrak{A}}'\mathfrak{A}$, and $\mathfrak{Z}=\tau Z$, $\mathfrak{W}=\tau W$, $\mathfrak{A}=\tau A$. Here we have

(24) $$K_0(Z, \overline{Z})=\text{det.}\,(E^{(2)}-\overline{\mathfrak{Z}}'\mathfrak{Z})\ ,$$

(25) $$(Z-A)'(Z-A)=\text{det.}\,(\mathfrak{Z}-\mathfrak{A})\ ,\qquad W'W=\text{det.}\,\mathfrak{W}\ ,$$

(26) $$K_0(Z, \overline{A})=\text{det.}\,(E^{(2)}-\overline{\mathfrak{A}}'\mathfrak{Z})\ ,$$

(27) $$E^{(2)}-\overline{\mathfrak{Z}}'\mathfrak{Z}=\overline{[\mathfrak{N}^{-1}(E^{(2)}-\overline{\mathfrak{A}}'\mathfrak{Z})]}'(E^{(2)}-\overline{\mathfrak{W}}'\mathfrak{W})[\mathfrak{N}^{-1}(E^{(2)}-\overline{\mathfrak{A}}'\mathfrak{Z})]\ .$$

Thus we see that the equalities (13), (17) and (18) hold for $n=4$, and consequently for any $n\geqq 4$, as is seen from (23)-(27).

11. Proof of Theorem 5 (II). The Jacobian of the transformation (21) at the point $Z=B=(b_1, b_2, b_3, b_4, 0, \cdots, 0)'$ is equal to

$$\left[\frac{\partial(\mathfrak{W})}{\partial(\mathfrak{Z})}\right]_{\mathfrak{Z}=\mathfrak{B}}\cdot K_0(A, \overline{A})^{(n-4)/2}K_0(B, \overline{A})^{-(n-4)}\ ,$$

where $\mathfrak{W}=\tau(w_1, w_2, w_3, w_4)'$, $\mathfrak{Z}=\tau(z_1, z_2, z_3, z_4)'$ and $\mathfrak{B}=\tau B_0=\tau(b_1, b_2, b_3, b_4)'$. Hence we have for (21)

$$\left[\frac{\partial(T_A(Z))}{\partial(Z)}\right]_{Z=B}=K_0(A, \overline{A})^{n/2}K_0(B, \overline{A})^{-n}\ ,$$

since the Jacobian of the transformation (23) is given in **6**, and is equal to $K_0(A, \overline{A})^2K_0(B, \overline{A})^{-4}$.

According to the consideration in **9** we have generally

(28) $$\frac{\partial(T_A(Z))}{\partial(Z)}=K_0(A, \overline{A})^{n/2}K_0(Z, \overline{A})^{-n}\ ,$$

for any points Z and A of $\mathfrak{M}_{(n)}$.

Hence we obtain

$$K(X, \overline{Y})=\frac{1}{v(\mathfrak{M}_{(n)})}K_0(X, \overline{Y})^{-n}\ ,$$

from (28) and (5). *Thus* (9) *is established.*

12. A generalization of Schwarz's lemma. Now we shall prove:

Theorem 7. *Any one-to-one analytic mapping of* $\mathfrak{M}_{(n)}$ *onto itself is of the form* $T_AT_{V,\theta}$.

To prove this it suffices to show that such a transformation leaving invariant zero vector is $T_{V,\theta}$.

Let Z be a matrix of type $(n, 1)$ and let us put

(29) $$N(Z)=\left\{\overline{Z}'Z+\sqrt{(\overline{Z}'Z)^2-|Z'Z|^2}\right\}^{1/2}\ ,$$

Lemma 2. *N has the norm-property, i.e.,*

$$N(Z_1+Z_2)\leqq N(Z_1)+N(Z_2)\,, \qquad N(\lambda Z)=|\lambda|\cdot N(Z) \tag{30}$$

for a complex number λ, and we have

$$\mathfrak{M}_{(n)}=\{Z|N(Z)<1\}\,.$$

For any two points A, B in $\mathfrak{M}_{(n)}$ we have

$$N(K)\leqq\frac{N(A)+N(B)}{1+N(A)N(B)}\,, \tag{31}$$

where $K=T_A(B)$. Here we note that $N(T_A(B))=N(T_B(A))$. These relations may be proved similarly as in **10** by reducing the problems to the case $n=4$. For the case $n=4$ we shall utilize the transformation τ which is defined for every vector of dimension 4 by (22). Then we have

$$N(Z_1-Z_2)=\|\mathfrak{Z}_1-\mathfrak{Z}_2\|$$

where $\mathfrak{Z}_j=\tau Z_j$, $j=1, 2$ and $\|\mathfrak{Z}\|$ means the norm of the square matrix $\mathfrak{Z}$. Therefore we have (30) by the property of the norm $\|\mathfrak{Z}\|$. The relation (31) can be proved from the corresponding relation for $\mathfrak{A}_{(2,2)}$ (cf. [5]).

Therefore, if we put

$$\rho^*(A, B)=\frac{1}{2}\log\frac{1+N(K)}{1-N(K)}\,,$$

where $K=T_A(B)$, we have

$$\rho^*(A, B)=\rho^*(B, A)$$

$$\rho^*(A, C)\leqq\rho^*(A, B)+\rho^*(B, C)$$

$$\rho^*(\sigma A, \sigma B)=\rho^*(A, B)$$

for any three points A, B, C of $\mathfrak{M}_{(n)}$ and for any analytic homeomorphism σ of $\mathfrak{M}_{(n)}$ onto itself.

Thus ρ^* is an invariant metric in $\mathfrak{M}_{(n)}$. Now Theorem 7 is a direct consequence of Theorem 8 below.

Theorem 8. *For any analytic mapping f of $\mathfrak{M}_{(n)}$ into itself we have*

$$\rho^*(f(X), f(Y))\leqq\rho^*(X, Y)\,.$$

Moreover, if the equality

$$\rho^*(f(Z_1), f(Z))=\rho^*(Z_1, Z)$$

holds for a point Z_1 and for every point Z in some neighbourhood of a point Z_0, then f is a homeomorphism of $\mathfrak{M}_{(n)}$ onto itself and is of the form $T_AT_{V,\theta}$.[5)]

5) A similar theorem as Theorem 8 was proved for $\mathfrak{A}_{(n,m)}$, $\mathfrak{S}_{(n)}$ by M. Sugawara [11] and the author [6], and foı $\mathfrak{L}_{(n)}$ by the author [7]. A more satisfactory formulation was given for the case of hyperspheres $\mathfrak{A}_{(n,1)}$ in the appendix of [7].

Proof. We have only to prove the theorem for a mapping f such that $f(0)=0$; the general case is immediately reduced to this special case. Let us put

$$W=T_{V,\theta}f(Z)=g(Z)=\begin{pmatrix} g_1(Z) \\ \vdots \\ g_n(Z) \end{pmatrix},$$

and

$$\varphi_j(Z)=g_{2j-1}(Z)+ig_{2j}(Z)$$
$$\psi_j(Z)=g_{2j-1}(Z)-ig_{2j}(Z)\ , \qquad j=1, 2, \cdots, m.$$

where $m=\left[\frac{n}{2}\right]$. Then we have

$$(32)\qquad \begin{cases} \sum_{j=1}^{m}|\varphi_j(Z)|^2+|g_{2m+1}(Z)|^2\leqq N(g(Z))^2\ . \\ \sum_{j=1}^{m}|\psi_j(Z)|^2+|g_{2m+1}(Z)|^2\leqq N(g(Z))^2\ . \end{cases}$$

Here these expressions are stated for the case $n=2m+1$ and the term $|g_{2m+1}(Z)|^2$ should be omitted in case $n=2m$; this remark will not be repeated in the following.

Let Z_0 be an arbitrary point of $\mathfrak{M}_{(n)}$ distinct from the origin. Then by Lemma 1 we have

$$(33)\qquad \begin{cases} \varphi_1(Z_0)=N(g(Z_0))=N(f(Z_0))\ , & \\ \varphi_j(Z_0)=0 & \text{for} \quad j>1, \\ \psi_j(Z_0)=0 & \text{for} \quad j>1, \\ g_{2m+1}(Z_0)=0 & \text{in case} \quad n=2m+1 \end{cases}$$

for a suitable $T_{V,\theta}$. If we consider a function

$$h(t)=\varphi_1\left(t\cdot\frac{1}{N(Z_0)}Z_0\right),$$

$h(t)$ is regular in the domain $|t|<1$ and $h(0)=0$, $|h(t)|<1$. Therefore we have $|g(t)|\leqq|t|$ by classical Schwarz's lemma and hence $N(f(Z_0))=|\varphi_1(Z_0)|\leqq N(Z_0)$. This proves the first part of the theorem.

To prove the second part, let f be an analytic mapping of $\mathfrak{M}_{(n)}$ into itself such that $f(0)=0$ and $N(f(Z))=N(Z)$ for every point Z of a neighbourhood $\mathfrak{U}$ of a point $Z_0\neq 0$. If we choose a suitable $T_{V,\theta}$, we have (33) for a point Z in $\mathfrak{U}$ and for the mapping f. Then we have further

$$\varphi_1\left(t\frac{1}{N(Z)}Z\right)=t$$

for every t such that $|t|<1$. From (32) we get

$$N\left(t\frac{1}{N(Z)}Z\right)=|t|=\left|\varphi_1\left(t\frac{1}{N(Z)}Z\right)\right|\leqq N\left(f\left(t\frac{1}{N(Z)}Z\right)\right).$$

This shows that

$$N(f(tZ))=N(tZ)=tN(Z) \qquad \text{for } |t|<\frac{1}{N(Z)} .$$

If we put

$$P_1(Z)=\lim_{t\to 0}\frac{f(tZ)}{t} ,$$

then the above consideration shows that

$$N(P_1(Z))=N(Z)$$

for every point Z in $\mathfrak{U}$, and $N(P_1(Z))\leqq N(Z)$ for any point Z in $\mathfrak{M}_{(n)}$.

Therefore by the same argument as in [7, pp. 54–55] the second part of the theorem is proved if we can show that a linear mapping f of $\mathfrak{M}_{(n)}$ into itself such that $f(0)=0$ and $N(f(Z))=N(Z)$ for every point Z in some neighbourhood of Z_0 is necessarily of the form $T_{V,\theta}$. Let f be such a mapping. Let

$$\Phi(\lambda;\,Z)=\lambda^2-2\lambda\overline{Z}'Z+|Z'Z|^2 .$$

Then we have $\Phi(\lambda;\,Z)=\Phi(\lambda;f(Z))$ by the similar argument as in the papers [6], [7], since $\Phi(\lambda;\,Z)$ is irreducible as a polynomial in λ with coefficients in $R(x_1, \cdots, x_n, y_1, \cdots, y_n)$ where $z_k=x_k+iy_k$ and R is the field of real numbers. Therefore we have

$$\overline{Z}'Z=\overline{f(Z)}'f(Z) , \qquad |Z'Z|=|f(Z)'f(Z)| .$$

Since f is linear we can write $f(Z)=UZ$ with a constant matrix U of order n. Each of the functions $f(Z)'f(Z)$ and $Z'Z$ are regular functions in $z_1, \cdots, z_n$ and $|f(Z)'f(Z)|=|Z'Z|$, and hence there exists a constant real number θ such that $f(Z)'f(Z)=e^{2i\theta}Z'Z$. We have therefore $U'U=e^{i\theta}E^{(n)}$. On the other hand, $\overline{U}'U=E^{(n)}$. Hence if we put $V=e^{-i\theta}U$, V is shown to be a real orthogonal matrix. Thus $f=e^{i\theta}VZ$, and the proof of the theorem is completed.

13. Proof of Theorem 6. If we express the transformation $T_AT_{V,\theta}$ in the form (12), we have

$$(34)\quad \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}=\frac{1}{K_0(A,\overline{A})^{1/2}}$$

$$\times\left(\begin{array}{c:cc} H_A+A\overline{A}'+\overline{A}A' & -(A+\overline{A}) & -i(A-\overline{A}) \\ \hdashline -(A'+\overline{A}') & 1+\frac{1}{2}(A'A+\overline{A}'A) & \frac{i}{2}(A'A-\overline{A'A}) \\ -i(A'-\overline{A}') & \frac{i}{2}(A'A-\overline{A'A}) & 1-\frac{1}{2}(A'A+\overline{A'A}) \end{array}\right)\left(\begin{array}{c:cc} V & \multicolumn{2}{c}{0} \\ \hdashline 0 & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{array}\right)$$

up to a factor ± 1. Hence det. $U_4=K_0(A,\overline{A})^{-1}(1-|A'A|^2)>0$.

On the other hand, since the set $\mathfrak{H}$ of all matrices of the form (34) induces the full group of analytical homeomorphisms of $\mathfrak{M}_{(n)}$ onto itself by Theorem 7, $\mathfrak{H}$ is a subgroup of index 2 in the group $\mathfrak{G}$ of all matrices U satisfying (11a) as is seen from the arguments in Siegel's lectures [8, §48]. Therefore any matrix U satisfying (11a) can be written in the form

$$K_A\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $K_A=U_{-A}$ and U_A is a matrix defined for $A\in\mathfrak{M}_{(n)}$ by (15) and T_1 and T_2 are real orthogonal matrices of order n and 2 respectively[6]; this factorization is unique since K_A is a positive definite real symmetric matrix. Moreover if U astisfies (11b), then we have det. $T_2=1$ and hence U must be of the form (34).

Thus any one-to-one analytic mapping of $\mathfrak{M}_{(n)}$ onto itself has the form (12) with real matrices U_1, U_2, U_3, U_4 satisfying the conditions (11a) and (11b), and *Theorem 6 is proved hereby*. The Jacobian of the transformation (12) is

$$\left[(1,\, i)\left\{U_3Z+U_4\begin{pmatrix} \frac{1}{2}(Z'Z+1) \\ \frac{i}{2}(Z'Z-1) \end{pmatrix}\right\}\right]^{-n}.$$

Remark. If we denote by $\mathfrak{H}_0$ the set of all orthogonal matrices contained in $\mathfrak{H}$, then $\mathfrak{H}_0$ is a maximal compact subgroup of $\mathfrak{H}$ and the correspondence

$$\Phi:\quad A\longrightarrow K_A\mathfrak{H}_0$$

gives a homeomorphism of $\mathfrak{M}_{(n)}$ onto the left coset space of $\mathfrak{H}$ modulo $\mathfrak{H}_0$ such that if $U\in\mathfrak{H}$ induces a transformation σ of $\mathfrak{M}_{(n)}$ onto itself then $\Phi(\sigma(A))=UK_A\mathfrak{H}_0$. The correspondence

$$\Psi:\quad A\longrightarrow K_A{}^2$$

gives a homeomorphism of $\mathfrak{M}_{(n)}$ onto the space consisting of all positive definite symmetric matrices of $\mathfrak{H}$ such that $\Psi(\sigma(A))=UK_A{}^2U'$ (for the case $\mathfrak{A}_{(n,m)}$ the positive definite Hermitian matrix corresponding to K_A is given by $\begin{pmatrix} N_A^{-1} & N_A^{-1}A \\ M_A^{-1}A' & M_A^{-1} \end{pmatrix}$, cf. [6]). (cf. Siegel [8]).

14. The invariant metrics. Let D be any irreducible symmetric domain given in **4**. Then the group of all one-to-one analytic mappings of D onto itself which leave the zero matrix invariant is irreducible as a group of linear transformations. From this it follows by a well-known lemma of 1. Schur that an Hermitian metric of D which is invariant under any one-to-one analytic transformation of D onto itself is unique up to a constant factor. Thus the Bergman metric

6) This expression can also be shown by the fact that the group $\mathfrak{G}$ induces the group of linear fractional transformations of real $\mathfrak{A}_{(n,2)}$ onto itself, our original proof was carried out by this method.

(35) $$ds^2 = 2\sum_{j,k} \frac{\partial^2 \log K_D(z, \bar{z})}{\partial z_j \partial \bar{z}_k} dz_j \overline{dz_k}$$

associated with the kernel function is essentially the unique invariant Hermite-Kähler metric for these domains, and is calculated as follows.

(36) $2(n+m)$ trace $[(E^{(m)} - \bar{Z}'Z)^{-1}\overline{dZ}'(E^{(n)} - Z\bar{Z}')^{-1}dZ]$, for $\mathfrak{R}_{(n,m)}$,

(37) $2(n+1)$ trace $[(E^{(n)} - \bar{Z}'Z)^{-1}\overline{dZ}'(E^{(n)} - Z\bar{Z}')^{-1}dZ]$, for $\mathfrak{S}_{(n)}$,

(38) $2(n-1)$ trace $[(E^{(n)} - \bar{Z}'Z)^{-1}\overline{dZ}'(E^{(n)} - Z\bar{Z}')^{-1}dZ]$, for $\mathfrak{L}_{(n)}$,

(39) $4nK_0(Z, \bar{Z})^{-2}\overline{dZ}'[K_0(Z, \bar{Z})(E^{(n)} - 2\bar{Z}Z') + 2(E^{(n)} - \bar{Z}Z')Z\bar{Z}'(E^{(n)} - \bar{Z}Z')]dZ$, for $\mathfrak{M}_{(n)}$.

The direct proofs for the invariance of the metric (36), (37) and (38) are already known. As for (39) we can proceed as follows: from (16)′ we see the invariance of the metric

$$ds^2 = K_0(Z, \bar{Z})^{-2}\overline{dZ}'(H_Z + Z\bar{Z}' - \bar{Z}Z')^2 dZ,$$

which is easily shown to be equal to (39) divided by $4n$.

The volume element derived from (35) is equal to $K_D(z, \bar{z})dv_z$ up to a positive constant factor, since D is homogeneous.

15. The Laplacian and harmonic functions. Let D be any bounded domain in p-dimensional complex Euclidean space. Then (35) can be written as follows:

$$ds^2 = (dz_1 \cdots dz_p d\bar{z}_1 \cdots d\bar{z}_p) G (dz_1 \cdots dz_p d\bar{z}_1 \cdots d\bar{z}_p),$$

where

$$G = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}, \qquad T = (T_{j\bar{k}}), \qquad T_{j\bar{k}} = \frac{\partial^2 \log K_D(z, \bar{z})}{\partial z_j \partial \bar{z}_k}.$$

Hence the first order differential parameter and Laplacian of the metric (35), when D is considered as a Riemannian space, are expressed as follows:

$$\Delta_1(\varphi, \psi) = \sum_{j=1}^{p} \sum_{k=1}^{p} \left(T^{k\bar{j}} \frac{\partial \varphi}{\partial z_j} \frac{\partial \psi}{\partial \bar{z}_k} + T^{\bar{k}j} \frac{\partial \varphi}{\partial \bar{z}_j} \frac{\partial \psi}{\partial z_k} \right),$$

$$\Delta \varphi = (\det T)^{-1} \left[\sum_{j=1}^{p} \sum_{k=1}^{p} \left\{ \frac{\partial}{\partial z_j} \left((\det T) T^{k\bar{j}} \frac{\partial \varphi}{\partial \bar{z}_k} \right) + \frac{\partial}{\partial \bar{z}_j} \left((\det T) T^{\bar{k}j} \frac{\partial \varphi}{\partial z_k} \right) \right\} \right],$$

where $T^{-1} = (T^{j\bar{k}})$ and $T^{\bar{k}j} = \overline{T^{k\bar{j}}}$.

If we denote by $A\binom{j\ l}{k\ m}$ the determinant of the minor matrix obtained from T by removing two rows j, l and two columns k, m, and by A_{jk} the cofactor of $T_{j\bar{k}}$ in T, then we have

$$\frac{\partial A_{jk}}{\partial z_s} = (-1)^{j+k} \sum_{l \neq j} \sum_{m \neq k} (-1)^{l+m} \varepsilon^{j\ k}_{l\ m} A\binom{j\ l}{k\ m} \frac{\partial T_{l\bar{m}}}{\partial z_s}.$$

where $\varepsilon^{j\,k}_{l\,m}$ means 1 or -1 according as $(j-l)(k-m)$ is positive or negative. Hence

$$\sum_{j=1}^{p}\frac{\partial A_{jk}}{\partial z_j}=(-1)^k\sum_{m\neq k}(-1)^m\sum_{j,l,j\neq l}(-1)^{j+l}\varepsilon^{j\,k}_{l\,m}A\binom{j\ l}{k\ m}\frac{\partial T_{l\bar{m}}}{\partial z_j}\ .$$

Since for $j\neq l$,

$$(-1)^{j+l}\varepsilon^{j\,k}_{l\,m}A\binom{j\ l}{k\ m}\frac{\partial T_{l\bar{m}}}{\partial z_j}+(-1)^{j+l}\varepsilon^{l\,k}_{j\,m}A\binom{j\ l}{k\ m}\frac{\partial T_{j\bar{m}}}{\partial z_l}=0\ ,$$

we have

$$\sum_{j=1}^{p}\frac{\partial A_{jk}}{\partial z_j}=0\ .$$

Therefore we obtain

Theorem 9. *The first order differential parameter and Laplacian for the metric* (35) *are given by*

$$\Delta_1(\varphi,\psi)=\sum_{j=1}^{p}\sum_{k=1}^{p}\left\{T^{k\bar{j}}\frac{\partial\varphi}{\partial z_j}\frac{\partial\psi}{\partial\bar{z}_k}+T^{\bar{k}j}\frac{\partial\varphi}{\partial\bar{z}_j}\frac{\partial\psi}{\partial z_k}\right\}, \tag{40}$$

$$\Delta\varphi=\sum_{j=1}^{p}\sum_{k=1}^{p}\left\{T^{k\bar{j}}\frac{\partial^2\varphi}{\partial z_j\partial\bar{z}_k}+T^{\bar{k}j}\frac{\partial^2\varphi}{\partial\bar{z}_j\partial z_k}\right\}, \tag{41}$$

and these are invariant under any analytical homeomorphism of D onto itself.[7]

The second part of the theorem follows readily from the invariance of the metric (35).

Let us put

$$r_D(x,y)=[\log(K_D(x,\bar{x})K_D(y,\bar{y})K_D(x,\bar{y})K_D(y,\bar{x}))]^{1/2}$$

for $x, y\in D$. Then

$$r_D(x,y)=r_D(y,x)>0 \quad \text{if} \quad x\neq y;\ r_D(x,x)=0\ , \tag{42}$$

$$r_D(x,y)=r_D(\sigma x,\sigma y) \quad \text{for any analytical homeomorphism } \sigma, \tag{43}$$

$$\frac{\partial^2 r_D{}^2(x,y)}{\partial x_j\partial\bar{x}_k}=\frac{\partial^2\log K_D(x,\bar{x})}{\partial x_j\partial\bar{x}_k}\ . \tag{44}$$

$r_D(x,y)$ is a distance function (not satisfying the triangle axiom) and satisfies the condition

$$\lim_{y\to x}\left\{\frac{r_D{}^2(x,y)}{\sum_{j,k}T_{j\bar{k}}(x)(y_j-x_j)(\bar{y}_k-\bar{x}_k)}\right\}=1\ . \tag{45}$$

From (41) and (44) we obtain

Theorem 10. *If we consider $r_D{}^2(x,y)$ as a function in x, then*

$$\Delta r_D{}^2(x,y)=2p\ , \tag{46}$$

$$\Delta\log K_D(x,\bar{x})=2p\ . \tag{47}$$

7) Thus formulae (41) holds for any Kähler metric.

Let us put

$$H(x, y)=\left(\frac{K_D(x, \bar{y})K_D(y, \bar{x})}{K_D(x, \bar{x})}\right)^q . \tag{48}$$

Then by (46) we have for a fixed $y \in \bar{D}$

$$\frac{1}{q}\left(-\frac{1}{H^2}\Delta_1(H, H)+\frac{1}{H}\Delta H\right)=-2p . \tag{49}$$

Here we shall consider H as a function in x.

For an analytical homeomorphism σ, if we put $x=\sigma\tilde{x}$, $y=\sigma\tilde{y}$, we have

$$H(x, y)=H(\sigma\tilde{x}, \sigma\tilde{y})=H(\tilde{x}, \tilde{y})\left|\frac{\partial\sigma(\tilde{y})}{\partial(\tilde{y})}\right|^{-2q} .$$

On the other hand

$$\Delta_x H(x, y)=\Delta_{\tilde{x}} H(\sigma\tilde{x}, \sigma\tilde{y})=[\Delta_{\tilde{x}} H(\tilde{x}, \tilde{y})]\left|\frac{\partial\sigma(\tilde{y})}{\partial(\tilde{y})}\right|^{-2q} .$$

Therefore

$$H(x, y)^{-1}\{\Delta_x H(x, y)\}=H(\tilde{x}, \tilde{y})^{-1}\{\Delta_{\tilde{x}} H(\tilde{x}, \tilde{y})\} . \tag{50}$$

If D is a circular domain with the origin $(0, \cdots, 0)$ as its centre, we have at $x=0$

$$H(x, y)^{-2}\{\Delta_1 H(x, y)\}=2q^2\sum_{j=1}^{p}\sum_{k=1}^{p} T_{j\bar{k}}(0)y_j\bar{y}_k , \tag{51}$$

where $y=(y_1, \cdots, y_p)$. To prove this, let $\{\varphi_\nu^{(1)}(x) \mid \nu=1, \cdots, p\}$ be the set of linear homogeneous functions belonging to the complete orthonormal system constructed in **3**, and let

$$\varphi_j{}^{(1)}(x)=\sum_{k=1}^{p} a_{jk}x_k , \qquad A=(a_{jk}) .$$

Then we have at $x=0$

$$\frac{1}{qH}\begin{pmatrix} \frac{\partial H}{\partial x_1} \\ \vdots \\ \frac{\partial H}{\partial x_p} \end{pmatrix}=v(D)\,A'\bar{A}\begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_p \end{pmatrix}, \qquad \frac{1}{qH}\begin{pmatrix} \frac{\partial H}{\partial \bar{x}_1} \\ \vdots \\ \frac{\partial H}{\partial \bar{x}_p} \end{pmatrix}=v(D)\bar{A}'A\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix},$$

$$(T_{j\bar{k}})=v(D)A'\bar{A} .$$

Hence we get (51).

Therefore we have at $x=0$

$$\frac{1}{H}\Delta H=2q\left\{q\sum_{j,k=1}^{p} T_{j\bar{k}}(0)y_j\bar{y}_k-p\right\} . \tag{52}$$

Let D be now one of the irreducible domains described in **4**. Let $\mathfrak{B}_0(D)$ be the set of all boundary points Y of D such that

(53) $$\bar{Y}'Y=E^{(m)}\,, \quad \text{in case} \quad D=\mathfrak{A}_{(n,m)}\,,$$

(54) $$\bar{Y}'Y=E^{(n)}\,, \quad \text{in case} \quad D=\gamma_{(n)}\,,$$

(55) $\bar{Y}'Y=E^{(n)}$ or the eigenvalues of $\bar{Y}'Y$ are all 1 except one which is zero according as n is even or odd, in case $D=\mathfrak{L}_{(n)}$,

(56) $$\bar{Y}'Y=1\,, \quad |Y'Y|=1\,, \quad \text{in case} \quad D=\mathfrak{M}_{(n)}\,.$$

Then it is easily shown that $\mathfrak{B}_0(D)$ is transformed onto itself by any analytical homeomorphism of D onto itself, and that the group $\mathfrak{G}_0(D)$ of all analytical homeomorphisms of D onto itself leaving the origin invariant is transitive on $\mathfrak{B}_0(D)$.

Let us put

(57) $$q_D=\begin{cases} \dfrac{n}{n+m}\,, & \text{in case} \quad D=\mathfrak{A}_{(n,m)}\,, \\ \dfrac{1}{2}\,, & \text{in case} \quad D=\mathfrak{S}_{(n)}\,, \\ \dfrac{1}{2} \text{ or } \dfrac{n}{2(n-1)}\,, & \text{according as } n=2m \text{ or } n=2m+1, \text{ in case } D=\mathfrak{L}_{(n)}\,, \\ \dfrac{1}{2}\,, & \text{in case} \quad D=\mathfrak{M}_{(n)}\,. \end{cases}$$

Then we obtain the following theorem from (52) and (36)–(39).

Theorem 11. *For $Y\in\mathfrak{B}_0(D)$ we have*

$$\Delta H_D(X,\,Y)=0\,,$$

that is, $H_D(X, Y)$ is a harmonic function in X for $X\in D$, where

$$H_D(X,\,Y)=v(D)^{q_D}\left(\frac{K_D(X,\,\bar{Y})K_D(Y,\,\bar{X})}{K_D(X,\,\bar{X})}\right)^{q_D}$$

and D is one of the irreducible domains $\mathfrak{A}_{(n,m)}$, $\mathfrak{S}_{(n)}$, $\mathfrak{L}_{(n)}$, $\mathfrak{M}_{(n)}$.

For the case $\mathfrak{A}_{(n,n)}$, J. Mitchell has proved Theorem 11 in a recent paper [12] by determining an explicit form of the Laplacian.

16. Poisson's integral.

Theorem 12. *Let a function $f(X)$ be regular in D and continuous on $\bar{D}$, and let D be one of the domains $\mathfrak{A}_{(n,n)}$, $\mathfrak{S}_{(n)}$, $\mathfrak{L}_{(2m)}$ and $\mathfrak{M}_{(n)}$. Then we have*

(58) $$f(X)=\int_{\mathfrak{B}_0(D)} H_D(X,\,Y)f(Y)d\mu_Y\,,$$

where $d\mu_Y$ means the Euclidean volume element for the set $\mathfrak{B}_0(D)$ divided by the total volume of $\mathfrak{B}_0(D)$.

J. Mitchell [12] proved this theorem for the case $D=\mathfrak{A}_{(n,n)}$ by using Cauchy's formula due to S. Bochner [13]. Here we shall proceed in a different way.

As is shown by Bochner [13] and Mitchell [12], we have

$$d\mu_Y=\begin{cases} c_n \det Y^{-n} dy_{11}dy_{12}\cdots dy_{21}\cdots dy_{nn}\,, & \text{for } D=\mathfrak{A}_{(n,n)}\,,\\ c_n^0 \det Y^{-(n+1)/2} dy_{11}dy_{12}\cdots dy_{1n}dy_{22}\cdots dy_{nn}\,, & \text{for } D=\gamma_{(n)}\,,\end{cases}$$

where

$$c_n=\frac{1!\,2!\cdots(n-1)!}{(2\pi i)^{n(n+1)/2}}\,,\qquad c_n^0=\frac{\Gamma\left(\frac{2}{2}\right)\Gamma\left(\frac{3}{2}\right)\cdots\Gamma\left(\frac{n+1}{2}\right)}{2^n\pi^{n(n+3)/2}i^{n(n+1)/2}}\,.$$

For the other cases we have

$$d\mu_Y=\begin{cases} c_n^1 \det Y^{-(n-1)/2} dy_{12}\cdots dy_{1n}dy_{23}\cdots dy_{n-1,n} & \text{for } D=\mathfrak{L}_{(n)},\ n=2m\,,\\ c_n^2 (Y'Y)^{-n/2} dy_1 dy_2\cdots dy_n & \text{for } D=\mathfrak{M}_{(n)}\,,\end{cases}$$

where C_n^1 and c_n^2 are non-zero constants such that

$$\int_{\mathfrak{B}_0(D)} d\mu_Y=1\,.$$

Then $d\mu_Y$ is invariant under any transformation of $\mathfrak{G}_0(D)$. We shall prove that $H_D(X, Y)d\mu_Y$ is invariant under any analytical homeomorphism σ of D onto itself, that is,

$$H_D(\sigma X, \sigma Y)d\mu_{\sigma(Y)}=H_D(X, Y)d\mu_Y\,. \tag{59}$$

In the following we shall restrict ourselves to the case $D=\mathfrak{M}_{(n)}$; the other cases can be treated similarly.

Let $Z\in\mathfrak{B}_0(\mathfrak{M}_{(n)})$. Then $|Z'Z|=1$ and hence

$$(Z-A)'(Z-A)=(Z'Z)K_0(A, \bar{Z})\,,\qquad \text{for } A\in\mathfrak{M}_{(n)},\ Z\in\mathfrak{B}_0(\mathfrak{M}_{(n)}) \tag{60}$$

and consequently we have from (17)

$$W'W=(Z'Z)K_0(A, \bar{Z})K_0(Z, \bar{A})^{-1}\,,\qquad W=T_A(Z) \tag{61}$$

for the transformation T_A defined by (16). (This shows at the same time that T_A carries $\mathfrak{B}_0(\mathfrak{M}_{(n)})$ onto itself.) Hence by (28) we obtain (59) immediately.

From (59) it follows that

$$\int_{\mathfrak{B}_0(D)} H_D(\sigma X, \sigma Y)f(\sigma Y)\,d\mu_{\sigma(Y)}=\int_{\mathfrak{B}_0(D)} H_D(X, Y)f(\sigma Y)d\mu_Y\,. \tag{62}$$

On the other hand, a monomial $y_1^{r_1}y_2^{r_2}\cdots y_n^{r_n}$ goes over into $-y_1^{r_1}y_2^{r_2}\cdots y_n^{r_n}$ by a transformation $Y\to e^{i\theta}Y$ with $\theta=\pi/\sum r_j$ where $\sum r_j>0$ and r_j are non-negative integers, and $d\mu_Y$ is invariant under this transformation. Therefore we have

$$\int_{\mathfrak{B}_0(D)} y_1^{r_1}y_2^{r_2}\cdots y_n^{r_n}d\mu_Y=0\,,\qquad \text{if } \sum r_j>0\,.$$

Since $\int_{\mathfrak{B}_0(D)} d\mu_Y = 1$ and by a theorem of H. Cartan a regular function on $\overline{D}$ can be expanded into a uniformly convergent (on $\overline{D}$) series of homogeneous polynomials, we see that if $f(X)$ is regular on $\overline{D}$, (58) holds for $X=0$, and consequently we obtain (58) for any $X \in D$ by virtue of (62).

In case $f(X)$ is regular in D and continuous on $\overline{D}$, we have therefore

$$f(tX) = \int_{\mathfrak{B}_0(D)} H_D(X, Y) f(tX) d\mu_Y$$

for any real number t such that $0 \leq t < 1$. Letting $t \to 1$, we see the validity of (58) for such a function $f(X)$. Thus Theorem 12 is proved.

Remark. In cases where the notation of square roots appears, the value is to be obtained by analytic continuation from a suitable initial value at a suitable point (cf. [13]).

17. The Cauchy formula of Bochner can be obtained from Theorem 12 by reversing Mitchell's argument. As an example, we take up the case $D = \mathfrak{A}_{(n,n)}$ which is treated by Bochner and Mitchell. In this case we can write (58) as follows:

$$(63) \qquad f(X) = c_n \int_{\mathfrak{B}_0(D)} \frac{f(Y) \det (E - \overline{X}'X)^n}{\det (E - \overline{X}'Y)^n \det (E - \overline{Y}'X)^n} \frac{dy_{11} \cdots dy_{nn}}{\det Y^n} .$$

If we put

$$\tilde{f}(Z) = f(Z) \det (E - \overline{X}'Z)^n \det (E - \overline{X}'X)^{-n}$$

for a point $X \in D$, then $\tilde{f}(Z)$ is regular in D and continuous on $\overline{D}$. Therefore we have by (63)

$$(64) \qquad f(X) = \tilde{f}(X) = c_n \int_{\mathfrak{B}_0(D)} \frac{f(Y)}{\det (Y - X)^n} dy_{11} dy_{12} \cdots dy_{nn} .$$

This is *the Cauchy formula due to Bochner* [13].

Similarly we have for a function f regular in D and continuous on $\overline{D}$, (cf. [13])

$$(65) \qquad f(X) = c_n^0 \int_{\mathfrak{B}_0(D)} \frac{f(Y)}{\det (Y - X)^{(n+1)/2}} dy_{11} dy_{12} \cdots dy_{nn} , \qquad \text{for } D = \mathfrak{S}_{(n)} ,$$

$$(66) \qquad f(X) = c_n^1 \int_{\mathfrak{B}_0(D)} \frac{f(Y)}{\det (Y - X)^{(n-1)/2}} dy_{12} \cdots dy_{n-1,n} , \qquad \text{for } D = \mathfrak{L}_{(n)}, n = 2m ;$$

$$(67) \qquad f(X) = c_n^2 \int_{\mathfrak{B}_0(D)} \frac{f(Y)}{[(Y - X)'(Y - X)]^{n/2}} dy_1 \cdots dy_n , \qquad \text{for } D = \mathfrak{M}_{(n)} .$$

Appendix

In a previous paper "On isometric transformations in spaces of matrices" (in Japanese with English summary), Rigaku (Science) vol. 3 (1948), we obtained the following results:

Let f be an isometric transformation in the space consisting of all matrices of type (n, m), $n \geqq m \geqq 2$ (case I), or of all symmetric matrices of order n (case II), or of all skew-symmetric matrices of order n (case III), or of all Hermitian matrices of order n (case V), where the distance between Z_1 and Z_2 is defined as the square root of the greatest eigenvalue of $(\bar{Z}_1-\bar{Z}_2)'(Z_1-Z_2)$. Then f is written in the following form:

I. $f(Z)=UZV+C$, $U\bar{Z}V+C$ (or $UZ'V+C$, $U\bar{Z}'V+C$ in case $n=m$)

II. $f(Z)=U'ZU+C$, $U'\bar{Z}U+C$;

III. $f(Z)=U'ZU+C$, $U'\bar{Z}U+C$ (or $U'Z^+U+C$, $U'\bar{Z}^+U+C$ in case $n=4$)

V. $f(Z)=U^{-1}ZU+C$, $-U^{-1}ZU+C$, $U^{-1}\bar{Z}U+C$, $-U^{-1}\bar{Z}U+C$,

as the case may be; here U, V are constant unitary matrices and C is also a constant matrix and Z^+ means the matrix obtained from Z by interchanging its $(1, 4)$-element with its $(2, 3)$-element.

Analogously to these results we can prove the following theorem.

Let f be a transformation of the set of all matrices of type $(n, 1)$ into itself such that $N(f(Z_1)-f(Z_2))=N(Z_1-Z_2)$, where $N(Z)$ is defined by (29). Then we have

IV. $f(Z)=e^{i\theta}VZ+C$, $e^{i\theta}V\bar{Z}+C$

where V is a constant real orthogonal matrix and θ is a constant real number.

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