# A Condition for the Metrizability of Topological Spaces and for $n$-dimensionality

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(Received November 5, 1954)

The known conditions for the metrizability of topological spaces are described generally in terms of open coverings. The following theorem, which will be established in the present note, is of some interest in view of this fact.

**Theorem 1.** *In order that a $T_1$-space $X$ be metrizable it is necessary and sufficient that there exist a countable collection $\{\mathfrak{F}_i\}$ of locally finite closed coverings of $X$ satisfying the condition:*

(1) *For any neighbourhood $U$ of any point $x$ of $X$ there exists some $i$ such that $S(x, \mathfrak{F}_i)\subset U$*[1].

As usual, by "$\dim X\leqq n$" we mean that every finite open covering of $X$ has an open refinement of order $\leqq n+1$. Then, in connection with Theorem 1 we obtain the following theorem.

**Theorem 2.** *In order that a $T_1$-space $X$ be a metrizable space with $\dim X\leqq n$, it is necessary and sufficient that there exist a countable collection $\{\mathfrak{F}_i\}$ of locally finite closed coverings of $X$ satisfying the condition* (1) *in Theorem 1 and the conditions* (2), (3) *and* (4) *below:*

(2) $\mathfrak{F}_i=\{F(\alpha_1, \cdots, \alpha_i)|\alpha_\nu\in\Omega,\ \nu=1, \cdots, i\}$, *where $F(\alpha_1, \cdots, \alpha_i)$ may be empty.*

(3) $F(\alpha_1, \cdots, \alpha_{i-1})=\bigcup\{F(\alpha_1, \cdots, \alpha_{i-1}, \gamma)|\gamma\in\Omega\}$.

(4) *The order of $\mathfrak{F}_i$ does not exceed $n+1$ for each $i$.*

An application of Theorem 1 and a related result of Theorem 2 will also be given.

**1.** To prove Theorem 1 we shall first state some lemmas.

**Lemma 1.** *If $\{F_\alpha\}$ is a locally finite closed covering of a $T_1$-space $X$, then $\{\operatorname{Int} F_\alpha\}$ is also a covering of $X$, where $\operatorname{Int} A$ means the interior of a set $A$ in $X$.*

**Lemma 2.** *If $\{\overline{G}_\alpha\}$ is a locally finite closed covering of a $T_1$-space $X$ and $G_\alpha$ are open sets of $X$, then there exists a closed covering $\{\overline{H}_\alpha\}$ of $X$ such that $H_\alpha$ are open sets and*

$$H_\alpha\subset G_\alpha, \quad \text{and} \quad H_\alpha\frown H_\beta=0 \quad \text{for} \quad \alpha\neq\beta .$$

**Lemma 3.** *Let $\{\overline{H}_\alpha\}$ be the covering described in Lemma 2. If we put $M_\alpha=\operatorname{Int}(\overline{H}_\alpha)$, then $\{\overline{M}_\alpha\}$ is a locally finite closed covering of $X$ such that $M_\alpha$ are regular open sets and $M_\alpha\frown M_\beta=0$ for $\alpha\neq\beta$.*

A locally finite closed covering of $X$ having the property described in Lemma

1) $S(x, \mathfrak{F}_i)$ means the union of the sets of $\mathfrak{F}_i$ containing the point $x$.

3 will be called a grating of $X$.

**Lemma 4.** *Let $\{\overline{M_\alpha}\}$ and $\{\overline{N_\beta}\}$ be two gratings of a $T_1$-space $X$. Then $\{\overline{M_\alpha \frown N_\beta}\}$ is also a grating of $X$.*

These lemmas can be proved easily (cf. [4]).

**2. Proof of Theorem 1.** Let $\{\mathfrak{F}_i\}$ be a countable collection of locally finite closed coverings of a $T_1$-space $X$ satisfying the condition (1) in Theorem 1.

By Lemmas 1, 2 and 3 there exists a grating $\overline{\mathfrak{M}}_i=\{\overline{M(\alpha, i)}|\alpha\in\Omega_i\}$ such that $\overline{\mathfrak{M}}_i$ is a refinement of $\mathfrak{F}_i$ for $i=1, 2, \ldots$ . Let us put

$$\overline{\mathfrak{W}}_i=\{\overline{W(\alpha_1, \cdots, \alpha_i)}|\alpha_\nu\in\Omega_\nu,\ \nu=1, 2, \cdots, i\}$$

where

$$W(\alpha_1, \cdots, \alpha_i)=\bigcap_{\nu=1}^{i} M(\alpha_\nu, \nu),\ \overline{M(\alpha_\nu, \nu)}\in\overline{\mathfrak{M}}_\nu .$$

Then by Lemma 4 $\overline{\mathfrak{W}}_i$ is a grating of $X$ and is a refinement of $\overline{\mathfrak{M}}_i$ and $\overline{\mathfrak{W}}_{i-1}$ for $i=1, 2, \ldots$ .

Let us put

$$V_n(x)=\mathrm{Int}\,(S(x, \overline{\mathfrak{W}}_n)) .$$

Then for any $V_n(x)$ we can find, by the condition (1), an positive integer $m=m(x, n)>n$ such that $S(x, \overline{\mathfrak{W}}_m)\subset V_n(x)$. We shall prove that if $V_m(y)\frown V_m(x)\neq 0$, then we have $V_m(y)\subset V_n(x)$ .

We first note that

$$y\in S(x, \overline{\mathfrak{W}}_m) ,$$

since otherwise we would have $\overline{W(\alpha_1, \cdots, \alpha_m)}\frown\overline{W(\beta_1, \cdots, \beta_m)}=0$ for any $\overline{W(\alpha_1, \cdots, \alpha_m)}\ni x$ and any $\overline{W(\beta_1, \cdots, \beta_m)}\ni y$, and hence $S(x, \overline{\mathfrak{W}}_m)\frown\overline{W(\beta_1, \cdots, \beta_m)}=0$, which would lead to a contradiction that $V_m(x)\frown V_m(y)=0$ .

Let $y\in\overline{W(\beta_1, \cdots, \beta_m)}$. Then we have

$$x\in\overline{W(\beta_1, \cdots, \beta_n)} ,$$

since if $x\notin\overline{W(\beta_1, \cdots, \beta_n)}$ we have $\overline{W(\alpha_1, \cdots, \alpha_n)}\frown\overline{W(\beta_1, \cdots, \beta_n)}=0$ for any $\overline{W(\alpha_1, \cdots, \alpha_n)}$ such that $x\in\overline{W(\alpha_1, \cdots, \alpha_n)}$, and hence $V_n(x)\frown\overline{W(\beta_1, \cdots, \beta_n)}=0$, which contradicts the fact that $y\in S(x, \overline{\mathfrak{W}}_m)\frown\overline{W(\beta_1, \cdots, \beta_m)}\subset V_n(x)\frown\overline{W(\beta_1, \cdots, \beta_n)}$.

Therefore we have $S(y, \overline{\mathfrak{W}}_m)\subset S(x, \overline{\mathfrak{W}}_n)$ and hence $V_m(y)\subset V_n(x)$. Thus we have proved that if $V_m(y)\frown V_m(x)\neq 0$, then $V_m(y)\subset V_n(x)$.

Since $\{V_n(x)\}$ is clearly a basis of neighbourhoods at $x$ and $V_{n+1}(x)\subset V_n(x)$ for $n=1, 2, \ldots$ , by a theorem of A. H. Frink [1] we see that $X$ is metrizable. This proves the "sufficient" part of Theorem 1. Since the "necessary" part is obvious by a theorem of A. H. Stone, Theorem 1 is completely proved.

**3.** As an application of Theorem 1 we shall prove the following theorem due to J. Nagata [8].

**Theorem 3.** *Let $\{A_\alpha\}$ be a locally finite closed covering of a $T_1$-space $X$. If each $A_\alpha$ is metrizable, then $X$ is also metrizable.*

Indeed, let $\{\mathfrak{F}_{i\alpha}\}$ be a countable collection of locally finite closed coverings of the subspace $A_\alpha$ such that it satisfies the condition (1) and $\mathfrak{F}_{i+1,\alpha}$ is a refinement of $\mathfrak{F}_{i\alpha}$ for $i=1, 2, \ldots$. If we put $\mathfrak{F}_i=\smile_\alpha \mathfrak{F}_{i\alpha}$, then $\mathfrak{F}_i$ is a locally finite closed covering of $X$ and $\{\mathfrak{F}_i\}$ satisfies the condition (1). Hence $X$ is metrizable by Theorem 1.

**4.** Let $\Omega$ de a non-empty set. For any two sequences $\alpha$, $\beta$ of elements from $\Omega$: $\alpha=(\alpha_1, \alpha_2, \cdots)$, $\beta=(\beta_1, \beta_2, \cdots)$ we define $\rho(\alpha, \beta)$ as follows:

$$\rho(\alpha, \beta)=\frac{1}{k}, \quad \text{if} \quad \alpha_i=\beta_i \quad \text{for } i<k \text{ and } \alpha_k\neq\beta_k,$$

$$\rho(\alpha, \beta)=0, \quad \text{if} \quad \alpha_i=\beta_i \quad \text{for } i=1, 2, \ldots .$$

The set of all sequences of elements from $\Omega$ determines a metric space by the distance function $\rho(\alpha, \beta)$; this space shall be denoted by $N(\Omega)$. We shall call $N(\Omega)$ a generalized Baire's zero-dimensional space, since $N(\Omega)$ is known as Baire's zero-dimensional space in case $\Omega$ consists of all the natural numbers. As is observed previously [6], $N(\Omega)$ is a complete metric space and $\dim N(\Omega)=0$.

**5.** We now turn to the proof of Theorem 2. The necessity of the condition of Theorem 2 is easily proved by repeated application of [6, Theorem 9.4] (cf. [2]). To prove the sufficiency of the condition of Theorem 2, let $\{\mathfrak{F}_i\}$ be a countable collection of locally finite closed coverings of $X$ satisfying the conditions (1) to (4). Then by Theorem 1, $X$ is metrizable.

The set of points $\alpha=(\alpha_1, \alpha_2, \cdots)$ of $N(\Omega)$ (cf. § 4) such that

$$\bigcap_{i=1}^{\infty} F(\alpha_1, \cdots, \alpha_i)\neq 0$$

shall be denoted by $P$. For any point $\alpha=(\alpha_1, \alpha_2, \cdots)$ of $P$ we put

$$f(\alpha)=\bigcap_{i=1}^{\infty} F(\alpha_1, \cdots, \alpha_i) .$$

It is seen that $f(\alpha)$ consists of only one point. Thus $f$ defines a single valued mapping of $P$ onto $X$. From (4) it follows that for each point $x$ of $X$ the inverse image $f^{-1}(x)$ consists of at most $n+1$ points. It is easily verified also that for $A\subset X$, $f^{-1}(A)$ is compact or separable according as $A$ is compact or separable. It is obvious that $f$ is continuous.

We shall show that the mapping $f$ is a closed mapping. For this purpose, let $A$ be any closed subset of $P$. We denote by $V(\alpha_1, \cdots, \alpha_i)$ the set of points $\gamma=(\gamma_1, \gamma_2, \cdots)$ of $N(\Omega)$ such tnat $\gamma_j=\alpha_j$ for $j=1, 2, \cdots, i$. Then $V(\alpha_1, \cdots, \alpha_i)$ is open and closed in $N(\Omega)$, and its diameter does not exceed $1/i$. Since

$$A=\smile\{A\frown V(\beta)|\beta\in\Omega\}, \ f(A\frown V(\beta))\subset F(\beta)$$

and $\mathfrak{F}_1=\{F(\beta)|\beta\}$ is locally finite, we have

$$\overline{f(A)}=\smile\{\overline{f(A\frown V(\beta))}|\beta\in\Omega\} .$$

Therefore for any given point $x_0$ of $\overline{f(A)}$ there exists some element $\alpha_1$ of $\Omega$ such that $x_0\in\overline{f(A\frown V(\alpha_1))}$. By an inductive process we can find a point $\alpha=(\alpha_1, \alpha_2, \cdots)$ of $N(\Omega)$ such that

$$x_0\in\overline{f(A\frown V(\alpha_1, \cdots, \alpha_i))}, \quad \text{for } i=1, 2, \cdots .$$

Hence we have

$$x_0 \in \bigcap_{i=1}^{\infty} \overline{f(A \frown V(\alpha_1, \cdots, \alpha_i))} \subset \bigcap_{i=1}^{\infty} F(\alpha_1. \cdots, \alpha_i) .$$

Therefore $x_0 = f(\alpha)$ and $\alpha = (\alpha_1, \alpha_2, \cdots) \in P$ .

On the other hand, $A \frown V(\alpha_1, \cdots, \alpha_i) \neq 0$. This shows that $\alpha \in \bar{A}$. Hence we have $x_0 = f(\alpha)$, $\alpha \in A$. This shows that $f(A)$ is closed. Therefore $f$ is a closed mapping.

Thus we have shown that if the condition of Theorem 2 holds then the condition of Theorem 4 below holds also, and furthermore we have proved the "only if" part of Theorem 4.

**Theorem 4.** *Let X be a metric space. Then we have* $\dim X \leqq n$ *if and only if there exist a subspace P of* $N(\Omega)$ *for suitable* $\Omega$ *and a closed continuous mapping f of P onto X such that for each point x of X the inverse image* $f^{-1}(x)$ *consists of at most* $n+1$ *points.*

**6.** Therefore, if we prove the "if" part of Theorem 4, Theorems 2 and 4 are proved completely. However, the "if" part of Theorem 4 follows readily from Theorem 5.

**Theorem 5.** *Let f be a closed continuous mapping of a metric space X onto another metric space Y such that for each point y of Y the inverse image* $f^{-1}(y)$ *consists of at most* $m+1$ *points. Then we have*

$$\dim Y \leqq \dim X + m .$$

This theorem is proved by W. Hurewicz for the case where $X$ and $Y$ are separable. In view of [6, Theorem 9.4] and [7, Theorem 2.6] or [6, Theorem 8.6], it is readily seen that the proof given in [5] or [3] remains valid for our general case with no or slight modification.

Adden in proof. As another application of Theorem 1 we have a simple proof of a theorem of S. Hanai (cf. his paper forthcoming in Proc. Japan Acad. 1955). Our results, combined with this theorem, yields at once the theorem: In order that a $T_1$-space $X$ be metrizable it is necessary and sufficient that there exist a subspace $P$ of $N(\Omega)$ for suitable $\Omega$ and a closed continuous mapping $f$ of $P$ onto $X$ such that $f^{-1}(x)$ is compact for each point $x$ of $X$.

## References

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