HANDBOOK OF
QUANTUM LOGIC
AND QUANTUM
STRUCTURES

Quantum Logic

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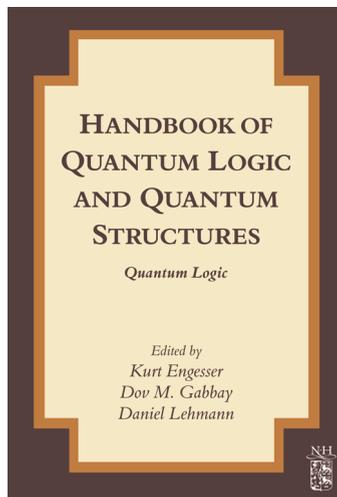
# GENTZEN METHODS IN QUANTUM LOGIC

Hirokazu Nishimura

1   INTRODUCTION

Since Birkhoff and von Neumann [Birkhoff and von Neumann, 1936] a new area of logical investigation has grown up under the name of quantum logic. During its early days emphasis was put exclusively on its algebraic aspects. A new impetus came from Dishkant and Goldblatt's ([Dishkant, 1972], [Dishkant, 1977] and [Goldblatt, 1974]) remarkable discovery on the relationship between ortholattices and the Brouwerian modal logic **B** in the 1970's, which is comparable to Mckinsey and Tarski's [McKinsey and Tarski, 1948] translation of intuitionistic logic into the modal logic **S**4. As the semantics of possible worlds has been one of the main tools in modal logic since Kripke [Kripke, 1963], the discovery naturally admitted to a Kripkian relational semantics of minimal quantum logic. Since it was then well known that there is a close relationship between Gentzen-style formulations of modal logics and their Kripkian relational semantics (cf. [Nishimura, 1983] and [Sato, 1977]), Nishimura [Nishimura, 1980] was driven on closing days of the 1970's to a Gentzen-style formulation of minimal quantum logic, which regrettably failed to enjoy the cut-elimination theorem. A more natural Gentzen-style formulation of minimal quantum logic with closer inspection on its relationship to the relational semantics was given by Cutland and Gibbins [Cutland and Gibbins, 1982], but it still failed to acquiesce in the cut-elimination property. The first cut-free Gentzen-style formulation of minimal quantum logic was presented by Tamura [Tamura, 1988], though it suffered from unnecessary clumsiness, which made his system appear more esoteric than it really was. A final step was taken again by Nishimura ([Nishimura, 1994a] and [Nishimura, 1994b]), which was followed by Takano's [Takano, 1995] significant remark that the inference rule from a sequent to its contraposition is redundant. The first stage of the story has thus ended, and the principal objective in this paper is to present its fruits to a novice thoroughly.

In Section 2 we will present our cut-free Gentzen-style sequential system **GMQL**. We will remark, following [Cutland and Gibbins, 1982], that admitting unrestricted (cut) as an inference rule would force our system **GMQL** to degenerated into classical logic. In Section 3 we will show, following [Cutland and Gibbins, 1982], that the inference rule from a sequent to its contraposition is admissible in **GMQL**. In Section 4 we will establish the fundamental fact that the negation ′ is involutive with respect to its proof-theoretical behaviors. In Section 5 the desired cut-elimination theorem is to be demonstrated. The final section is devoted to the

completeness theorem with respect to the relational semantics of Dishkant and Goldblatt.

The reader may wonder what is to be the second stage of the story. We will give two suggestions. The modal logics **S**4 and **B** stand to the modal logic **S**5 in opposite directions, but they are complementary against **S**5, as may be illustrated in the following figure:

$$\boxed{\textbf{S}5}$$
$$\nearrow \qquad \nwarrow$$
$$\boxed{\textbf{S}4} \qquad\qquad \boxed{\textbf{B}}$$
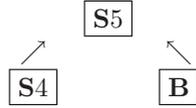
The complementarity of the modal logics **S**4 and **B** corresponds to the following complementarity of intuitionistic logic and minimal quantum logic against classical logic, as may be illustrated in the following figure:

$$\boxed{\text{classical logic}}$$
$$\nearrow \qquad\qquad \nwarrow$$
$$\boxed{\text{intuitionistic logic}} \qquad\qquad \boxed{\text{minimal quantum logic}}$$
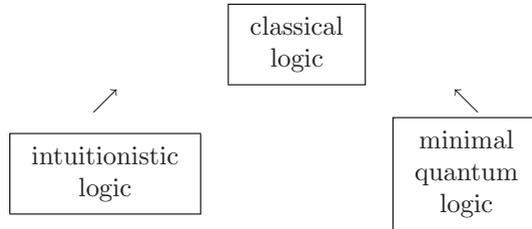
Logics between classical logic and intuitionistic logic have been studied vigorously under the name of intermediate logics. It would be interesting to investigate logics between classical logic and minimal quantum logic, among which you can find quantum logic.

The other intriguing topic for future study is a semantical proof of the cut-elimination theorem of **GMQL**. In other words, it would be interesting to give a proof of the completeness theorem with respect to the Kripkian relational semantics without any recourse to the cut-elimination theorem, which would surely open a new area of research.

2   MINIMAL QUANTUM LOGIC IN GENTZEN STYLE

The sequential system **GMQL** that we have enunciated for minimal quantum logic in our [Nishimura, 1994a] and that has then been elaborated by Takano in [Takano, 1995] consists of the following inference rules:

$$\frac{\Gamma \to \Delta}{\pi, \Gamma \to \Delta, \Sigma} \quad \text{(extension)}$$

$$\frac{\alpha, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta} \quad \frac{\beta, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta} \quad (\wedge \to)$$

$$\frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \alpha \vee \beta} \quad \frac{\Gamma \to \Delta, \beta}{\Gamma \to \Delta, \alpha \vee \beta} \quad (\to \vee)$$

$$\frac{\alpha \to \Delta \quad \beta \to \Delta}{\alpha \vee \beta \to \Delta} \quad (\vee \to)$$

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta} \quad (\to \wedge)$$

$$\frac{\Gamma \to \Delta}{\Delta', \Gamma \to} \quad ('\to)$$

$$\frac{\Gamma \to \Delta}{\to \Delta, \Gamma'} \quad (\to')$$

$$\frac{\alpha, \Gamma \to \Delta}{\alpha'', \Gamma \to \Delta} \quad (''\to)$$

$$\frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \alpha''} \quad (\to'')$$

$$\frac{\alpha', \Gamma \to \Delta}{(\alpha \vee \beta)', \Gamma \to \Delta} \quad \frac{\beta', \Gamma \to \Delta}{(\alpha \vee \beta)', \Gamma \to \Delta} \quad (\vee' \to)$$

$$\frac{\Gamma \to \Delta, \alpha'}{\Gamma \to \Delta, (\alpha \wedge \beta)'} \quad \frac{\Gamma \to \Delta, \beta'}{\Gamma \to \Delta, (\alpha \wedge \beta)'} \quad (\to \wedge')$$

$$\frac{\alpha' \to \Delta \quad \beta' \to \Delta}{(\alpha \wedge \beta)' \to \Delta,} \quad (\wedge' \to)$$

$$\frac{\Gamma \to \alpha' \quad \Gamma \to \beta'}{\Gamma \to (\alpha \vee \beta)'} \quad (\to \vee')$$

$$\frac{\Gamma \to \alpha' \quad \Gamma \to \beta'}{\alpha \vee \beta, \Gamma \to} \quad (\vee \to')$$

$$\frac{\alpha' \to \Delta \quad \beta' \to \Delta}{\to \Delta, \alpha \wedge \beta} \quad ('\to \wedge)$$

Now some notational and terminological comments are in order. In this paper we adopt $'$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction) as primitive logical symbols. Propositional variables are denoted by $p, q, \dots$, while wffs (well-formed formulas), also called formulas, are denoted by $\alpha, \beta, \dots$. The *grade of* a wff α, denoted by $\mathcal{G}(\alpha)$, is defined inductively as follows:

1. $\mathcal{G}(p) = 0$ for any propositional variable p.

2. $\mathcal{G}(\alpha') = \mathcal{G}(\alpha) + 1$.

3. $\mathcal{G}(\alpha \wedge \beta) = \mathcal{G}(\alpha \vee \beta) = \mathcal{G}(\alpha) + \mathcal{G}(\beta) + 2$.

Finite (possibly empty) sets of wffs are denoted by $\Gamma, \Delta, \Pi, \dots$. Given a finite set Γ of wffs, Γ' denotes the set $\{\alpha' | \alpha \in \Gamma\}$. A *sequent* $\Gamma \to \Delta$ means the ordered pair (Γ, Δ) of finite sets Γ and Δ of wffs, while the sets Γ and Δ are called the *antecedent* and the *succedent of* the sequent $\Gamma \to \Delta$, respectively. Such self-explanatory notations as $\Pi, \Gamma \to \Delta, \Sigma$ for $\Pi \cup \Gamma \to \Delta \cup \Sigma$ are used freely. A sequent of the form $\alpha \to \alpha$ is called an *axiom sequent*. Given a sequent $\Gamma \to \Delta$, the sequent $\Delta' \to \Gamma'$ is called the *contraposition* of $\Gamma \to \Delta$.

The notion of a *proof* P of a sequent $\Gamma \to \Delta$ with *length* n is defined inductively as follows:

1. Any axiom sequent $\alpha \to \alpha$ is a proof of itself with length 0.

2. If P is a proof of a sequent $\Gamma \to \Delta$ with length n and

$$\frac{\Gamma \to \Delta}{\Pi \to \Sigma}$$

   is an instance of an inference rule of **GMQL**, then

$$\frac{P}{\Pi \to \Sigma}$$

   is a proof of the sequent $\Pi \to \Sigma$ with length $n + 1$.

3. If P_i is a proof of a sequent $\Gamma_i \to \Delta_i$ with length n_i $(i = 1, 2)$ and

$$\frac{\Gamma_1 \to \Delta_1 \quad \Gamma_2 \to \Delta_2}{\Pi \to \Sigma}$$

   is an instance of an inference rule of **GMQL**, then

$$\frac{P_1 \quad P_2}{\Pi \to \Sigma}$$

   is a proof of the sequent $\Pi \to \Sigma$ with length $\max\{n_1, n_2\} + 1$.

The length of a proof P is denoted by $l(P)$. A sequent $\Gamma \to \Delta$ is said to be *provable* if it has a proof. Otherwise it is called *consistent*.

Although our cut-free sequential system **GMQL** does not satisfy the so-called subformula property in its strict sense, it gives a decision procedure for the word problem of free ortholattices once the completeness theorem is established, for which it suffices to note that $\mathcal{G}(\alpha') < \mathcal{G}((\alpha \wedge \beta)')$ and $\mathcal{G}(\beta') < \mathcal{G}((\alpha \wedge \beta)')$ for the rule $(\wedge' \to)$ by way of example. For algebraic and semantical decision procedures, the reader is referred to [Bruns, 1976], [Goldblatt, 1974] and [Goldblatt, 1975]. Fortunately, minimal quantum logic enjoys these three kinds of decision procedures. However, algebraic and semantical approaches to the decision problem of quantum logic have not succeeded so far. This is why we should try the third one.

Generally speaking, (cut) is the inference rule of the following form:

$$\frac{\Gamma_1 \rightarrow \Delta_1, \alpha \quad \alpha, \Gamma_2 \rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2} \quad \text{(cut)}$$

However, following [Cutland and Gibbins, 1982], we should remark that the inference rule (cut) in such an unrestricted form forces our system **GMQL** to degenerate into classical logic. In other words, we have

PROPOSITION 1. *If we add the inference rule (cut) to the system **GMQL**, then we obtain classical logic. Schematically, we have*

$$\textbf{\textit{GMQL}}\textit{+(cut) = classical logic}$$

Proof. It suffices to show that the following three rules are admissible in **GMQL**+(cut):

$$\frac{\Gamma \rightarrow \Delta, \alpha}{\alpha', \Gamma \rightarrow \Delta} \quad (\prime \rightarrow)_c$$

$$\frac{\alpha, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \alpha'} \quad (\rightarrow')_c$$

$$\frac{\Gamma \rightarrow \Delta, \alpha \quad \Gamma \rightarrow \Delta, \beta}{\Gamma \rightarrow \Delta, \alpha \wedge \beta} \quad (\rightarrow \wedge)_c$$

Since we have

$$\frac{\overline{\Gamma \rightarrow \Delta, \alpha} \quad \dfrac{\alpha \rightarrow \alpha}{\alpha, \alpha' \rightarrow} \ (\prime\rightarrow)}{\alpha', \Gamma \rightarrow \Delta} \quad \text{(cut)}$$

the inference rule $(\prime \rightarrow)_c$ is admissible in **GMQL**+(cut). Similarly, since we have

$$\frac{\dfrac{\alpha \rightarrow \alpha}{\rightarrow \alpha', \alpha} \ (\rightarrow') \quad \alpha, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \alpha'} \quad \text{(cut)}$$

the inference rule $(\rightarrow')_c$ is admissible in **GMQL**+(cut). Now we deal with the last inference rule $(\rightarrow \wedge)_c$. The sequents $\Gamma, \Delta' \rightarrow \alpha$ and $\Gamma, \Delta' \rightarrow \beta$ follow from the sequents $\Gamma \rightarrow \Delta, \alpha$ and $\Gamma \rightarrow \Delta, \beta$ respectively by a finite number of applications of the inference rule $(\prime \rightarrow)_c$. Now we have

$$\frac{\Gamma, \Delta' \rightarrow \alpha \quad \Gamma, \Delta' \rightarrow \beta}{\Gamma, \Delta' \rightarrow \alpha \wedge \beta} \quad (\rightarrow \wedge)$$

The sequent $\Gamma \rightarrow \Delta'', \alpha \wedge \beta$ follows from the sequent $\Gamma, \Delta' \rightarrow \alpha \wedge \beta$ by a finite number of applications of the inference rule $(\rightarrow')_c$. Since we have

$$\frac{\dfrac{\gamma \rightarrow \gamma}{\rightarrow \gamma, \gamma'} \ (\rightarrow')}{\gamma'' \rightarrow \gamma} \quad (\prime\rightarrow)_c$$

we are sure that the sequent $\gamma'' \to \gamma$ is provable in **GMQL**+(cut) for any $\gamma \in \Delta$. Therefore the desired sequent $\Gamma \to \Delta, \alpha \wedge \beta$ follows from the sequent $\Gamma \to \Delta'', \alpha \wedge \beta$ by a finite number of applications of the inference rule (cut). ∎

This is the reason why Cutland and Gibbins [Cutland and Gibbins, 1982] should have proposed (cut) in the following restricted form:

$$\frac{\Gamma \to \Delta_1, \alpha \quad \alpha \to \Delta_2}{\Gamma \to \Delta_1, \Delta_2} \quad \text{(cut-1)}$$

$$\frac{\Gamma_1 \to \alpha \quad \alpha, \Gamma_2 \to \Delta}{\Gamma_1, \Gamma_2 \to \Delta} \quad \text{(cut-2)}$$

The wff α in (cut-1) and (cut-2) is called the *cut formula*. Both (cut-1) and (cut-2) are called $(\text{cut})_q$ as a whole. Roughly speaking, if we deprive our system **GMQL** of the inference rules $(\vee' \to)$, $(\to \wedge')$, $(\wedge' \to)$, $(\to \vee')$, $(\vee \to')$, and $('\to \wedge)$ and we agree to admit the inference rules (cut-1) and (cut-2), then we obtain the system of Cutland and Gibbins [Cutland and Gibbins, 1982]. We will prove in Section 5 that the inference rules (cut-1) and (cut-2) are admissible in **GMQL**.

Tamura [Tamura, 1988] gave a cut-free system by exploiting the legacy of Cutland and Gibbins [Cutland and Gibbins, 1982] but incorporating their inference rules surely except (cut-1) and (cut-2) into his system in an unnecessarily restricted manner. This unreasonable restriction forced him in the proof of the cut-elimination theorem to combine wffs in the antecedent of a sequent by conjunction and wffs in its succedent by disjunction, and then to dissolve such unnatural combinations. Such a proof is not compatible with Gentzen's [Gentzen, 1935] original philosophy and aesthetics, and is to be avoided if possible. Furthermore, the conceptual significance of Lemma 4 in Tamura's [Tamura, 1988] paper remained vague at best there. This is distilled into the duality theorem in Section 4, which is followed by the so-called cut-elimination theorem in Section 5.

Our original **GMQL**, proposed in [Nishimura, 1994a], contains the following inference rule besides the above ones:

$$\frac{\Gamma \to \Delta}{\Delta' \to \Gamma'} \quad ('\to')$$

It was pointed out by Takano [Takano, 1995] that the rule is redundant, which is the topic of the succeeding section.

3   THE CONTRAPOSITION THEOREM

The principal objective in this section is to show the following theorem on the lines of Takano [Takano, 1995].

THEOREM 2. *The following inference rule is admissible in* **GMQL**.

$$\frac{\Gamma \to \Delta}{\Delta' \to \Gamma'} \quad ('\to')$$

To establish the above theorem, we introduce an auxiliary formal system to be denoted by $\mathbf{GMQL}^{\#}$ and to be obtained from $\mathbf{GMQL}$ by admitting not only sequents $\alpha \to \alpha$ but also sequents $\alpha, \alpha' \to$ and $\to \alpha, \alpha'$ as axiom sequents while deleting the inference rules $(' \to)$ and $(\to')$ and adding the following two inference rules:

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{(\alpha \wedge \beta)', \Gamma \to} \quad (\wedge' \to)^{\#}$$

$$\frac{\alpha \to \Delta \quad \beta \to \Delta}{\to \Delta, (\alpha \vee \beta)'} (\to \vee')^{\#}$$

We need three lemmas so as to establish the equivalence of $\mathbf{GMQL}$ and $\mathbf{GMQL}^{\#}$.

LEMMA 3.

1. *If a sequent $\alpha'', \Gamma \to \Delta$ is provable in $\mathbf{GMQL}^{\#}$, then so is $\alpha, \Gamma \to \Delta$.*

2. *If a sequent $\Gamma \to \Delta, \alpha''$ is provable in $\mathbf{GMQL}^{\#}$, then so is $\Gamma \to \Delta, \alpha$.*

Proof. We prove only the first statement by induction on the length $l(P)$ of a proof P of the sequent $\alpha'', \Gamma \to \Delta$, while leaving a similar treatment of the second statement to the reader. Our treatment is divided into several cases, some of which are again divided into several subcases.

1. The case that the sequent $\alpha'', \Gamma \to \Delta$ is an axiom sequent: We divide this case into three subcases.

    (a) The subcase that the sequent $\alpha'', \Gamma \to \Delta$ is $\alpha'' \to \alpha''$: Since we have

    $$\frac{\alpha \to \alpha}{\alpha \to \alpha''} \quad (\to'')$$

    the sequent $\alpha \to \alpha''$ is also provable.

    (b) The subcase that the sequent $\alpha'', \Gamma \to \Delta$ is $\alpha'', \alpha' \to$: The sequent $\alpha, \alpha' \to$ is an axiom, and so is provable.

    (c) The subcase that the sequent $\alpha'', \Gamma \to \Delta$ is $\alpha'', \alpha''' \to$: Since we have

    $$\frac{\alpha, \alpha' \to}{\alpha, \alpha''' \to} \quad ('' \to)$$

    the sequent $\alpha, \alpha''' \to$ is also provable.

2. The case that the last inference is (extension): The last inference has one of the following two forms:

$$\frac{\Gamma_1 \to \Delta_1}{\alpha'', \Gamma_2, \Gamma_1 \to \Delta_1, \Delta_2} \quad (\text{extension})$$

$$\frac{\alpha'', \Gamma_1 \to \Delta_1}{\alpha'', \Gamma_2, \Gamma_1 \to \Delta_1, \Delta_2} \quad \text{(extension)}$$

In the former case, since we have

$$\frac{\Gamma_1 \to \Delta_1}{\alpha, \Gamma_2, \Gamma_1 \to \Delta_1, \Delta_2} \quad \text{(extension)}$$

the sequent $\alpha, \Gamma_2, \Gamma_1 \to \Delta_1, \Delta_2$ is also provable. In the latter case, since the sequent $\alpha, \Gamma_1 \to \Delta_1$ is provable by induction hypothesis and we have

$$\frac{\alpha, \Gamma_1 \to \Delta_1}{\alpha, \Gamma_2, \Gamma_1 \to \Delta_1, \Delta_2} \quad \text{(extension)}$$

the sequent $\alpha, \Gamma_2, \Gamma_1 \to \Delta_1, \Delta_2$ is also provable.

3. The case that the last inference is $('' \to)$: We divide this case into two subcases according as the principal formula of the last inference is α'' or not.

   (a) The subcase that the principal formula of the last inference is α'': The last inference has one of the following two forms:

   $$\frac{\alpha, \Gamma \to \Delta}{\alpha'', \Gamma \to \Delta} \quad ('' \to)$$

   $$\frac{\alpha'', \alpha, \Gamma \to \Delta}{\alpha'', \Gamma \to \Delta} \quad ('' \to)$$

   In the former case the sequent $\alpha, \Gamma \to \Delta$ is palpably provable, while in the latter case it should be provable by induction hypothesis.

   (b) The subcase that the principal formula of the last inference is not α'': The last inference has the form

   $$\frac{\alpha'', \beta, \Gamma_1 \to \Delta}{\alpha'', \beta'', \Gamma_1 \to \Delta} \quad ('' \to)$$

   Since the sequent $\alpha, \beta, \Gamma_1 \to \Delta$ is provable by induction hypothesis and we have
   $$\frac{\alpha, \beta, \Gamma_1 \to \Delta}{\alpha, \beta'', \Gamma_1 \to \Delta} \quad ('' \to)$$

   the sequent $\alpha, \beta'', \Gamma_1 \to \Delta$ is also provable.

4. The case that the last inference is neither (extension) nor $('' \to)$:Similar to the subcase (3-b).

■

LEMMA 4. *The inference rule $(' \to')$ is admissible in $\mathbf{GMQL}^{\#}$.*

Proof. We will prove that if a sequent $\Gamma \to \Delta$ is provable in $\mathbf{GMQL}^{\#}$, then its contraposition $\Delta' \to \Gamma'$ is also provable in $\mathbf{GMQL}^{\#}$. The proof is carried out by induction on the length of a proof P of the given sequent $\Gamma \to \Delta$. Our treatment is divided into several cases.

1. The case that the given sequent $\Gamma \to \Delta$ is an axiom sequent: The sequent $\Gamma \to \Delta$ has one of the following three forms $\alpha \to \alpha$, $\alpha', \alpha \to$ and $\to \alpha, \alpha'$, whose contrapositions are also axioms $\alpha' \to \alpha'$, $\to \alpha', \alpha''$ and $\alpha'', \alpha' \to$.

2. The case that the last inference in P is (extension), $(\wedge \to)$, $(\to \wedge)$, $(\vee \to)$, $(\to \vee)$, $('' \to)$ or $(\to '')$: All these cases can be dealt with similarly, so we deal only with the case that the last inference is $(\to \wedge)$ as follows:

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta} \quad (\to \wedge)$$

Since the sequents $\alpha' \to \Gamma'$ and $\beta' \to \Gamma'$ are provable by induction hypothesis and we have

$$\frac{\alpha' \to \Gamma' \quad \beta' \to \Gamma'}{(\alpha \wedge \beta)' \to \Gamma'} \quad (\wedge' \to)$$

the sequent $(\alpha \wedge \beta)' \to \Gamma'$ is also provable.

3. The case that the last inference in P is either $(\wedge' \to)^{\#}$ or $(\to \vee')^{\#}$: Here we deal only with the former case, leaving a similar treatment of the latter case to the reader. So we suppose that the last inference in P is

$$\frac{\Gamma_1 \to \alpha \quad \Gamma_1 \to \beta}{(\alpha \wedge \beta)', \Gamma_1 \to} \quad (\wedge' \to)^{\#}$$

Since the sequents $\alpha' \to \Gamma_1'$ and $\beta' \to \Gamma_1'$ are provable by induction hypothesis and we have

$$\frac{\dfrac{\alpha' \to \Gamma_1' \quad \beta' \to \Gamma_1'}{\to \Gamma_1', \alpha \wedge \beta} \quad (' \to \wedge)}{\to \Gamma_1', (\alpha \wedge \beta)''} \quad (\to '')$$

we are sure that the sequent $\to \Gamma_1', (\alpha \wedge \beta)''$ is also provable.

4. The case that the last inference in P is either $(' \to \wedge)$ or $(\vee \to ')$: Here we deal only with the former case, leaving a similar treatment of the latter case to the reader. So we suppose that the last inference in P is

$$\frac{\alpha' \to \Delta_1 \quad \beta' \to \Delta_1}{\to \Delta_1, \alpha \wedge \beta} \quad (' \to \wedge)$$

The sequents $\Delta_1' \to \alpha''$ and $\Delta_1' \to \beta''$ are provable by induction hypothesis, which imply by Lemma 3 that the sequents $\Delta_1' \to \alpha$ and $\Delta_1' \to \beta$ are also provable. Since we have

$$\frac{\Delta_1' \to \alpha \quad \Delta_1' \to \beta}{(\alpha \wedge \beta)', \Delta_1' \to} \quad (\wedge' \to)^{\#}$$

we are sure that the sequent $(\alpha \wedge \beta)', \Delta_1' \to$ is also provable.

5. The case that the last inference in P is $(\wedge' \to)$, $(\to \wedge')$, $(\vee' \to)$ or $(\to \vee')$: Here we deal only with the first case, leaving similar treatments of the remaining three cases to the reader. So we suppose that the last inference in P is

$$\frac{\alpha' \to \Delta_1 \quad \beta' \to \Delta_1}{(\alpha \wedge \beta)' \to \Delta_1} \ \ (\wedge' \to)$$

The sequents $\Delta_1' \to \alpha''$ and $\Delta_1' \to \beta''$ are provable by induction hypothesis, which imply by Lemma 3 that the sequents $\Delta_1' \to \alpha$ and $\Delta_1' \to \beta$ are also provable. Since we have

$$\frac{\dfrac{\Delta_1' \to \alpha \quad \Delta_1' \to \beta}{\Delta_1' \to \alpha \wedge \beta} \ \ (\to \wedge)}{\Delta_1' \to (\alpha \wedge \beta)''} \ \ (\to'')$$

we are sure that the sequent $\Delta_1' \to (\alpha \wedge \beta)''$ is also provable.

∎

LEMMA 5.

1. *If a sequent $\Gamma \to \Delta$ is provable in $\textbf{GMQL}^{\#}$, then so is $\Delta', \Gamma \to$.*

2. *If a sequent $\Gamma \to \Delta$ is provable in $\textbf{GMQL}^{\#}$, then so is $\to \Delta, \Gamma'$.*

Proof. The proof is by induction on the length of a proof P of the given sequent $\Gamma \to \Delta$. We deal only with the first statement, leaving a similar treatment of the second treatment to the reader. Our treatment is divided into several cases.

1. The case that the given sequent $\Gamma \to \Delta$ is an axiom sequent: The sequent $\Gamma \to \Delta$ is one of the three forms $\alpha \to \alpha$, $\alpha', \alpha \to$ and $\to \alpha, \alpha'$. Then the sequent $\Delta', \Gamma \to$ is one of the two forms $\alpha', \alpha \to$ and $\alpha'', \alpha' \to$, both of which are axioms.

2. The case that the last inference in P is (extension), $(\wedge \to)$, $(\to \vee)$, $('' \to)$, $(\to'')$ or $(\vee' \to)$: Here we deal only with the third case, leaving similar treatments of the remaining five cases to the reader. Thus the last inference of P is of the following form:

$$\frac{\Gamma \to \Delta_1, \alpha}{\Gamma \to \Delta_1, \alpha \vee \beta} \ \ (\to \vee)$$

Since the sequent $\alpha', \Delta_1', \Gamma \to$ is provable by induction hypothesis and we have

$$\frac{\alpha', \Delta_1', \Gamma \to}{(\alpha \vee \beta)', \Delta_1', \Gamma \to} \ \ (\vee' \to)$$

we are sure that the sequent $(\alpha \vee \beta)', \Delta_1', \Gamma \to$ is also provable.

3. The case that the last inference in P is $(\to \wedge')$: The last inference in P is in the following form:

$$\frac{\Gamma \to \Delta_1, \alpha'}{\Gamma \to \Delta_1, (\alpha \wedge \beta)'} \quad (\to \wedge')$$

The sequent $\alpha'', \Delta_1', \Gamma \to$ is provable by induction hypothesis, which implies by Lemma 3 that the sequent $\alpha, \Delta_1', \Gamma \to$ is also provable. Since we have

$$\frac{\dfrac{\alpha, \Delta_1', \Gamma \to}{\alpha \wedge \beta, \Delta_1', \Gamma \to} \ (\wedge \to)}{(\alpha \wedge \beta)'', \Delta_1', \Gamma \to} \quad ('' \to)$$

we are sure that the sequent $(\alpha \wedge \beta)'', \Delta_1', \Gamma \to$ is also provable.

4. The case that the last inference in P is $(\to \wedge)$: The last inference in P is of the following form:

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta} \quad (\to \wedge)$$

Since we have

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{(\alpha \wedge \beta)', \Gamma \to} \quad (\wedge' \to)^{\#}$$

we are sure that the sequent $(\alpha \wedge \beta)', \Gamma \to$ is also provable.

5. The case that the last inference in P is $(\to \vee')$: The last inference in P is of the following form:

$$\frac{\Gamma \to \alpha' \quad \Gamma \to \beta'}{\Gamma \to (\alpha \vee \beta)'} \quad (\to \vee')$$

Since we have

$$\frac{\dfrac{\Gamma \to \alpha' \quad \Gamma \to \beta'}{\alpha \vee \beta, \Gamma \to} \ (\vee \to')}{(\alpha \vee \beta)'', \Gamma \to} \quad ('' \to)$$

we are sure that the sequent $(\alpha \vee \beta)'', \Gamma \to$ is also provable.

6. The case that the last inference in P is either $(' \to \wedge)$ or $(\wedge' \to)$: The last inference in P is one of the following two forms:

$$\frac{\alpha' \to \Delta_1 \quad \beta' \to \Delta_1}{\to \Delta_1, \alpha \wedge \beta} \quad (' \to \wedge)$$

$$\frac{\alpha' \to \Delta_1 \quad \beta' \to \Delta_1}{(\alpha \wedge \beta)' \to \Delta_1} \quad (\wedge' \to)$$

In both cases, the sequents $\Delta_1' \to \alpha''$ and $\Delta_1' \to \beta''$ are provable by Lemma 4, which implies by dint of Lemma 3 that the sequents $\Delta_1' \to \alpha$ and $\Delta_1' \to \beta$ are also provable. Since we have

$$\frac{\Delta_1' \to \alpha \quad \Delta_1' \to \beta}{(\alpha \wedge \beta)', \Delta_1' \to} \quad (\wedge' \to)^{\#}$$

we are sure that the sequent $(\alpha \vee \beta)'', \Gamma \to$ is also provable.

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7. The case that the last inference in P is $(\vee \to)$: The last inference in P is of the following form:

$$\frac{\alpha \to \Delta \quad \beta \to \Delta}{\alpha \vee \beta \to \Delta} \quad (\vee \to)$$

The sequents $\Delta' \to \alpha'$ and $\Delta' \to \beta'$ are provable by Lemma 4. Since we have

$$\frac{\Delta' \to \alpha' \quad \Delta' \to \beta'}{\Delta', \alpha \vee \beta \to} \quad (\vee \to')$$

we are sure that the sequent $\Delta', \alpha \vee \beta \to$ is also provable.

8. The case that the last inference in P is $(\to \vee')^{\#}$: The last inference in P is of the following form:

$$\frac{\alpha \to \Delta \quad \beta \to \Delta}{\to \Delta, (\alpha \vee \beta)'} (\to \vee')^{\#}$$

The sequents $\Delta' \to \alpha'$ and $\Delta' \to \beta'$ are provable by Lemma 4. Since we have

$$\frac{\dfrac{\Delta' \to \alpha' \quad \Delta' \to \beta'}{\Delta', \alpha \vee \beta \to} \quad (\vee \to')}{\Delta', (\alpha \vee \beta)'' \to} \quad ('' \to)$$

9. The case that the last inference in P is either $(\vee \to')$ or $(\wedge' \to)^{\#}$: There is nothing to prove, for the succedent Δ of the given sequent $\Gamma \to \Delta$ is empty.

∎

Now we are ready to present a proof of the main theorem.

THEOREM 6. *A sequent* $\Gamma \to \Delta$ *is provable in* $\mathbf{GMQL}^{\#}$ *iff it is provable in* $\mathbf{GMQL}$.

Proof.

1. First we deal with the only-if part. Since

$$\frac{\alpha \to \alpha}{\alpha', \alpha \to} \quad (' \to)$$

and

$$\frac{\alpha \to \alpha}{\to \alpha, \alpha'} \quad (\to ')$$

sequents $\alpha, \alpha' \to$ and $\to \alpha, \alpha'$ are provable in $\mathbf{GMQL}$. Since

$$\frac{\dfrac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta} \quad (\to \wedge)}{(\alpha \wedge \beta)', \Gamma \to} \quad (' \to)$$

and

$$\frac{\dfrac{\alpha \to \Delta \qquad \beta \to \Delta}{\alpha \wedge \beta \to \Delta} \quad (\vee \to)}{\to \Delta, (\alpha \vee \beta)'} \quad (\to')$$

the inferene rules $(\wedge' \to)^{\#}$ and $(\to \vee')^{\#}$ are admissible in **GMQL**. Thus the only-if part has been established

2. The if part follows directly from Lemmas 4 and 5

∎

Our desired theorem at the beginning of this section follows at once from the above theorem. Since the inference rule $('\to')$ is admissible in **GMQL**, we will often take it as a basic inference rule of the system **GMQL**.

4   THE DUALITY THEOREM

Two wffs α and β are said to be *provably equivalent*, in notation $\alpha \simeq \beta$, if for any finite sets Γ and Δ of wffs we have that

1. the sequent $\alpha, \Gamma \to \Delta$ is provable iff the sequent $\beta, \Gamma \to \Delta$ is provable; and

2. the sequent $\Gamma \to \Delta, \alpha$ is provable iff the sequent $\Gamma \to \Delta, \beta$ is provable.

It is easy to see that this is indeed an equivalence relation among wffs. We will show that it is even a congruence relation.

THEOREM 7.   *(The fundamental theorem of provability equivalence). If $\alpha_1 \simeq \beta_1$ and $\alpha_2 \simeq \beta_2$, then $\alpha_1' \sim \beta_1', \alpha_1 \wedge \alpha_2 \simeq \beta_1 \wedge \beta_2$, and $\alpha_1 \vee \alpha_2 \simeq \beta_1 \vee \beta_2$.*

Proof. If $\gamma, \delta_1, ..., \delta_n$ are wffs and $p_1, ..., p_n$ are distinct propositional variables, we write $\gamma[\delta_1/p_1, ..., \delta_n/p_n]$ for the wff obtained from γ by replacing every occurrence of p_i by δ_i $(1 \leq i \leq n)$. Whenever we use this notation, it will always be assumed that the propositional variables at issue are distinct. The theorem follows readily from the following two statements:

1. If $\delta_1 \simeq \sigma_1, ..., \delta_n \simeq \sigma_n$ and a sequent $\gamma[\delta_1/p_1, ..., \delta_n/p_n], \Gamma \to \Delta$ has a proof P with $l(P) \leq m$, then the sequent $\gamma[\sigma_1/p_1, ..., \sigma_n/p_n], \Gamma \to \Delta$ is also provable.

2. If $\delta_1 \simeq \sigma_1, ..., \delta_n \simeq \sigma_n$ and a sequent $\Gamma \to \Delta, \gamma[\delta_1/p_1, ..., \delta_n/p_n]$ has a proof P with $l(P) \leq m$, then the sequent $\Gamma \to \Delta, \gamma[\sigma_1/p_1, ..., \sigma_n/p_n]$ is also provable.

These two statements are proved simultaneously by double induction principally on $\mathcal{G}(\gamma)$ and secondly on m. The proof is divided into cases according to which inference rule is used as the last inference in P. The details are safely left to the reader. ∎

THEOREM 8. *(The first duality theorem). If $\alpha \simeq \beta$, then $\alpha \simeq \beta''$.*

Proof. It suffices to show the following claim:

CLAIM 9.

1. If a sequent $\alpha, \Gamma \to \Delta$ is provable, then the sequent $\beta'', \Gamma \to \Delta$ is also provable.

2. If a sequent $\Gamma \to \Delta, \alpha$ is provable, then the sequent $\Gamma \to \Delta, \beta''$ is also provable.

3. If a sequent $\alpha'', \Gamma \to \Delta$ is provable, then the sequent $\beta, \Gamma \to \Delta$ is also provable.

4. If a sequent $\Gamma \to \Delta, \alpha''$ is provable, then the sequent $\Gamma \to \Delta, \beta$ is also provable.

It is easy to see that the first and second statements of the above claim follow at once from a simple application of the inference rules $('' \to)$ and $(\to '')$, respectively, while 3 and 4 of the above claim follow at once from the following, ostensibly more general statement.

CLAIM 10. If $\alpha_1 \simeq \beta_1, ..., \alpha_n \simeq \beta_n, \alpha_{n+1} \simeq \beta_{n+1}, ..., \alpha_{n+m} \simeq \beta_{n+m}$ and a sequent $\alpha_1'', ..., \alpha_n'', \Gamma \to \Delta, \alpha_{n+1}'', ..., \alpha_{n+m}''$ has a proof P with $l(P) \leq k$ then the sequent $\beta_1, ..., \beta_n, \Gamma \to \Delta, \beta_{n+1}, ..., \beta_{n+m}$ is also provable.

We will prove Claim 10 by induction on k. The proof is divided into cases according to which inference rule is used in the last step of P. To make the notation simpler, we proceed as if $n = 1$ and $m = 0$, leaving safely easy but due modifications to the reader. In dealing with the rules $(\wedge \to), (\to \vee), (\vee' \to)$ and $(\to \wedge')$, each of which consists of two forms, we treat only one of them.

1. The case that the sequent $\alpha_1'', \Gamma \to \Delta$ is an axiom sequent: It must be that $\alpha_1'' \to \alpha_1''$. Since $\beta_1 \to \beta_1$ is an axiom sequent and $\alpha_1 \simeq \beta_1$ by assumption, the sequent $\beta_1 \to \alpha_1$ is provable, which implies that the sequent $\beta_1 \to \alpha_1''$ is also provable as follows:

$$\frac{\beta_1 \to \alpha_1}{\beta_1 \to \alpha_1''} \quad (\to '')$$

2. The case that the last inference of the proof of the sequent $\alpha_1'', \Gamma \to \Delta$ is (extension), $(\wedge \to), (\to \vee), (\wedge \to), (\to ''), (\vee' \to), (\to \wedge'), (\to \vee')$, or $(\vee \to')$: All the cases can be dealt with similarly, so here we deal only with the case in which the last inference of the proof is $(\to \wedge)$ as follows:

$$\frac{\alpha_1'', \Gamma \to \beta \quad \alpha_1'', \Gamma \to \gamma}{\alpha_1'', \Gamma \to \beta \wedge \gamma} \quad (\to \wedge)$$

By the induction hypothesis the sequents $\beta_1, \Gamma \to \beta$ and $\beta_1, \Gamma \to \gamma$ are provable, which gives the desired result as follows:

$$\frac{\beta_1, \Gamma \to \beta \qquad \beta_1, \Gamma \to \gamma}{\beta_1, \Gamma \to \beta \wedge \gamma} \qquad (\to \wedge)$$

3. The case that the last inference of the proof of $\alpha_1'', \Gamma \to \Delta$ is $('' \to)$: Then the last inference is one of the following two forms.

$$\frac{\alpha_1, \Gamma \to \Delta}{\alpha_1'', \Gamma \to \Delta} \qquad ('' \to \wedge) \qquad\qquad \frac{\alpha_1'', \beta'', \Gamma_1 \to \Delta}{\alpha_1'', \beta'', \Gamma_1 \to \Delta} \qquad ('' \to)$$

In the former case the sequent $\beta_1, \Gamma \to \Delta$ is provable for $\alpha_1 \simeq \beta_1$ and the sequent $\alpha_1, \Gamma \to \Delta$ is provable by assumption. In the latter case the sequent $\beta_1, \beta, \Gamma_1 \to \Delta$ is provable by the induction hypothesis, which implies that the sequent $\beta_1, \beta'', \Gamma_1 \to \Delta$ is provable as follows:

$$\frac{\beta_1, \beta, \Gamma_1 \to \Delta}{\beta_1, \beta'', \Gamma_1 \to \Delta} \qquad ('' \to)$$

4. The case that the last inference of the proof of the sequent $\alpha_1'', \Gamma \to \Delta$ is $(' \to)$: This case is divided into several subcases according to how the upper sequent of $(' \to)$ is obtained.

   (a) The case that the upper sequent of $(' \to)$ is an axiom sequent: In this case the axiom sequent must be $\alpha_1' \to \alpha_1'$, so the proof that we must consider is as follows:

   $$\frac{\alpha_1' \to \alpha_1'}{\alpha_1'', \alpha_1' \to} \qquad (' \to)$$

   Since the sequent $\alpha_1 \to \alpha_1$ is an axiom sequent and $\alpha_1 \simeq \beta_1$ by assumption, the sequent $\beta_1 \to \alpha_1$ is provable, which implies that the desired sequent $\beta_1, \alpha_1' \to$ is also provable as follows:

   $$\frac{\beta_1, \alpha_1 \to}{\alpha_1', \beta_1 \to} \qquad (' \to)$$

   (b) The case that the upper sequent of $(' \to)$ is obtained as the lower sequent of (extension), $(\wedge \to), ('' \to)$, or $(\vee' \to)$: All these cases can be dealt with similarly, so here we consider only the case of $('' \to)$, in which the last two steps of the proof go as follows:

   $$\frac{\dfrac{\beta, \Gamma_2 \to \alpha_1', \Gamma_1}{\beta'', \Gamma_2 \to \alpha_1', \Gamma_1} \quad ('' \to)}{\alpha_1'', \Gamma_1', \beta'', \Gamma_2 \to} \qquad (' \to)$$

The sequent $\alpha_1'', \Gamma_1', \beta, \Gamma_2 \to$ has a shorter proof than the sequent $\alpha_1'', \Gamma_1', \beta'', \Gamma_2 \to$, as follows:

$$\frac{\beta, \Gamma_2, \to \alpha_1', \Gamma_1'}{\alpha_1'', \Gamma_1', \beta, \Gamma_2 \to} \quad ('\to)$$

Therefore the sequent $\beta_1, \Gamma_1', \beta, \Gamma_2 \to$ is provable by the induction hypothesis, which implies that the desired sequent $\beta_1, \Gamma_1', \beta'', \Gamma_2 \to$ is also provable as follows:

$$\frac{\beta_1, \Gamma_1', \beta, \Gamma_2 \to}{\beta_1, \Gamma_1', \beta'', \Gamma_2 \to} \quad (''\to)$$

(c) The case that the upper sequent of $('\to)$ is obtained as the lower sequent of $(\to'')$: The last two steps of the proof that we must consider can be supposed to be one of the following two forms:

$$\frac{\dfrac{\Gamma_2 \to \beta, \Gamma_1}{\Gamma_2 \to \beta'', \Gamma_1} \quad (\to'')}{\beta'', \Gamma_1, \Gamma_2 \to} \quad ('\to)$$

$$\frac{\dfrac{\Gamma_2 \to \alpha_1', \beta, \Gamma_1}{\Gamma_2 \to \alpha_1', \beta'', \Gamma_1} \quad (\to'')}{\alpha_1'', \beta''', \Gamma_1, \Gamma_2 \to} \quad ('\to)$$

In the former case α_1 is supposed to be β', Since the latter case can be dealt with in a similar manner to the case (2), here we deal with the former case, in which the sequent $\alpha_1, \Gamma_1', \Gamma_2 \to$ is provable with a shorter proof than that of the sequent $\alpha_1'', \Gamma_1', \Gamma_2 \to$ as follows:

$$\frac{\Gamma_2, \to \beta, \Gamma_1}{\beta', \Gamma_1', \Gamma_2 \to} \quad ('\to)$$

Thus the desired sequent $\beta_1, \Gamma_2', \Gamma_2 \to$ is also provable by hypothesis,

(d) The case that the upper sequent of $('\to)$ is obtained as the lower sequent of $(\to \vee)$: The last two steps of the proof go as follows:

$$\frac{\dfrac{\Gamma_2 \to \alpha_1', \beta, \Gamma_1}{\Gamma_2 \to \alpha_1', \beta \vee \gamma, \Gamma_1} \quad (\to \vee)}{\alpha_1'', (\beta \vee \gamma)', \Gamma_1', \Gamma_2 \to} \quad ('\to)$$

The sequent $\alpha_1'', \beta', \Gamma_1', \Gamma_2 \to$ has a shorter proof than the sequent $\alpha_1'', (\beta \vee \gamma)', \Gamma_1', \Gamma_2 \to$ as follows:

$$\frac{\Gamma_2, \to \alpha_1', \beta, \Gamma_1}{\alpha_1'', \beta', \Gamma_1', \Gamma_2 \to} \quad ('\to)$$

Therefore the sequent $\beta_1, \beta', \Gamma'_1, \Gamma_2 \rightarrow$ is provable by the induction hypothesis, which implies that the desired sequent $\beta_1, (\beta \vee \gamma)', \Gamma'_1, \Gamma_2 \rightarrow$ is provable as follows:

$$\frac{\beta_1, \beta', \Gamma'_1, \Gamma_2 \rightarrow}{\beta_1, (\beta \vee \gamma)', \Gamma'_1, \Gamma_2 \rightarrow} \quad (\vee' \rightarrow)$$

(e) The case that the upper sequent of $('\rightarrow)$ is obtained as the lower sequent of $(\rightarrow \wedge')$; The last two steps of the proof are of one of the following two forms:

$$\frac{\dfrac{\Gamma_2 \rightarrow \alpha'_1, \beta', \Gamma_1}{\Gamma_2 \rightarrow \alpha'_1, (\beta \wedge \gamma)', \Gamma_1} \quad (\rightarrow \wedge')}{\alpha''_1, (\beta \wedge \gamma)'', \Gamma'_1, \Gamma_2 \rightarrow} \quad ('\rightarrow)$$

$$\frac{\dfrac{\Gamma_2 \rightarrow \beta', \Gamma_1}{\Gamma_2 \rightarrow (\beta \wedge \gamma)', \Gamma_1} \quad (\rightarrow \wedge')}{(\beta \wedge \gamma)'', \Gamma'_1, \Gamma_2 \rightarrow} \quad ('\rightarrow)$$

In the latter case α_1 is assumed to be $\beta \wedge \gamma$. Here we deal only with the former case, leaving a similar treatment of the latter case to the reader. The sequent $\alpha''_1, \beta'', \Gamma'_1, \Gamma_2 \rightarrow$ has a shorter proof than the sequent $\alpha''_1, (\beta \wedge \gamma)'', \Gamma'_1, \Gamma_2 \rightarrow$ as follows:

$$\frac{\Gamma_2, \rightarrow \alpha'_1, \beta', \Gamma_1}{\alpha''_1, \beta'', \Gamma'_1, \Gamma_2 \rightarrow} \quad ('\rightarrow)$$

This implies by the induction hypothesis that the sequent $\beta_1, \beta, \Gamma'_1, \Gamma_2 \rightarrow$ is also provable. Thus the desired sequent $\beta_1, (\beta \wedge \gamma)'', \Gamma'_1, \Gamma_2 \rightarrow$ is also provable, as follows:

$$\frac{\dfrac{\beta_1, \beta, \Gamma'_1, \Gamma_2 \rightarrow}{\beta_1, \beta \wedge \gamma, \Gamma'_1, \Gamma_2 \rightarrow} \quad (\wedge \rightarrow)}{\beta_1, (\beta \wedge \gamma)'', \Gamma'_1, \Gamma_2 \rightarrow} \quad ('' \rightarrow)$$

(f) The case that the upper sequent of $('\rightarrow)$ is obtained as the lower sequent of $(\vee \rightarrow)$: The last two steps of the proof that we must consider go as follows:

$$\frac{\dfrac{\beta \rightarrow \alpha'_1, \Gamma_1 \qquad \gamma \rightarrow \alpha'_1, \Gamma_1}{\beta \vee \gamma \rightarrow \alpha'_1, \Gamma_1} \quad (\vee \rightarrow)}{\alpha''_1, \Gamma'_1, \beta \vee \gamma \rightarrow} \quad ('\rightarrow)$$

The sequents $\alpha''_1, \Gamma'_1 \rightarrow \beta'$ and $\alpha''_1, \Gamma'_1 \rightarrow \gamma'$ are provable with shorter proofs than that of $\alpha''_1, \Gamma'_1, \beta \vee \gamma \rightarrow$ as follows

$$\frac{\beta \rightarrow \alpha'_1, \Gamma_1}{\alpha''_1, \Gamma'_1 \rightarrow \beta'} \quad ('\rightarrow')$$

$$\frac{\gamma \to \alpha'_1, \Gamma_1}{\alpha''_1, \Gamma'_1 \to \gamma'} \quad ('\to')$$

Therefore the sequents $\beta_1, \Gamma'_1 \to \beta'$ and $\beta_1, \Gamma'_1 \to \gamma'$ are provable by the induction hypothesis, which implies that the desired sequent $\beta \vee \gamma, \beta_1, \Gamma'_1 \to$ is also provable as follows:

$$\frac{\beta_1, \Gamma'_1 \to \beta' \quad \beta_1, \Gamma'_1 \to \gamma'}{\beta \vee \gamma, \beta_1, \Gamma'_1 \to} \quad (\vee \to')$$

(g) The case that the upper sequent of $('\to)$ is obtained as the lower sequent of $(\wedge' \to)$: The last two steps of the proof that we must consider go as follows:

$$\frac{\dfrac{\beta' \to \alpha'_1, \Gamma_1 \quad \gamma' \to \alpha'_1, \Gamma_1}{(\beta \wedge \gamma)' \to \alpha_1, \Gamma_1} \quad (\wedge' \to)}{\alpha''_1, \Gamma'_1, (\beta \vee \gamma)' \to} \quad ('\to)$$

The sequents $\alpha''_1, \Gamma'_1 \to \beta''$ and $\alpha''_1, \Gamma'_1 \to \gamma''$ are provable with shorter proofs than that of the sequent $\alpha''_1, \Gamma'_1, (\beta \vee \gamma)' \to$ as follows:

$$\frac{\beta' \to \alpha'_1, \Gamma_1}{\alpha''_1, \Gamma'_1 \to \beta''} \quad ('\to')$$

$$\frac{\gamma' \to \alpha'_1, \Gamma_1}{\alpha''_1, \Gamma'_1 \to \gamma''} \quad ('\to')$$

Thus the sequents $\beta_1, \Gamma'_1 \to \beta$ and $\beta_1, \Gamma'_1 \to \gamma$ are provable by the induction hypothesis, which implies that the desired sequent $(\beta \wedge \gamma)', \beta_1, \Gamma'_1 \to$ is also provable as follows:

$$\frac{\dfrac{\beta_1, \Gamma'_1 \to \beta \quad \beta_1, \Gamma'_1 \to \gamma}{\beta_1, \Gamma'_1 \to \beta \wedge \gamma,} \quad (\to \wedge)}{(\beta \wedge \gamma)', \beta_1, \Gamma'_1 \to} \quad ('\to)$$

(h) The case that the upper sequent of $('\to)$ is obtained as the lower sequent of $(\to \vee')$: The last two steps of the proof that we must consider go as follows:

$$\frac{\dfrac{\Gamma'_1 \to \beta' \quad \Gamma_1 \to \gamma'}{\Gamma'_1 \to (\beta \vee \gamma)'} \quad (\to \vee')}{(\beta \vee \gamma)'', \Gamma_1 \to} \quad ('\to)$$

Here α_1 is supposed to be $\beta \vee \gamma$. The sequent $\beta \vee \gamma, \Gamma_1 \to$ is provable as follows:

$$\frac{\Gamma_1 \to \beta' \quad \Gamma_1 \to \gamma'}{\beta \vee \gamma, \Gamma_1 \to} \quad (\vee \to')$$

Since $\beta_1 \simeq \alpha_1$ by assumption, the desired sequent $\beta_1, \Gamma_1 \to$ is also provable.

(i) The case that the upper sequent of $('\to)$ is obtained as the lower sequent of $('\to \wedge)$: The last two steps of the proof that we must consider go as follows:

$$\frac{\dfrac{\beta' \to \alpha_1', \Gamma_1' \qquad \gamma' \to \alpha_1', \Gamma_1'}{\to \beta \wedge \gamma, \alpha_1', \Gamma_1'} \quad ('\to \wedge)}{(\beta \wedge \gamma)', \alpha_1'', \Gamma_1' \to} \quad ('\to)$$

The sequents $\alpha_1', \Gamma_1' \to \beta$ and $\alpha_1'', \Gamma_1' \to \gamma''$ are provable with shorter proofs than that of the sequent $(\beta \wedge \gamma)', \alpha_1'', \Gamma_1' \to$ as follows:

$$\frac{\beta' \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \to \beta''} \qquad ('\to')$$

$$\frac{\gamma' \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \to \gamma''} \qquad ('\to')$$

Thus the sequents $\beta_1, \Gamma_1' \to \beta$ and $\beta_1, \Gamma_1' \to \gamma$ are provable by the induction hypothesis, which implies that the desired sequent $(\beta \wedge \gamma)', \beta_1, \Gamma_1' \to$ is also provable as follows:

$$\frac{\dfrac{\beta_1, \Gamma_1' \to \beta \qquad \beta_1, \Gamma_1' \to \gamma}{\beta_1, \Gamma_1' \to \beta \wedge \gamma} \quad (\to \wedge)}{(\beta \wedge \gamma)', \beta_1, \Gamma_1' \to} \quad ('\to)$$

(j) The case that the upper sequent of $('\to)$ is obtained as the lower sequent of $('\to')$: The last two steps of the proof that we must consider go as follows:

$$\frac{\dfrac{\alpha_1, \Gamma_1 \to \Gamma_2}{\Gamma_2' \to \alpha_1', \Gamma_1'} \quad ('\to')}{\alpha_1'', \Gamma_1'', \Gamma_2' \to} \quad ('\to)$$

Since the sequent $\alpha_1, \Gamma_1 \to \Gamma_2$ is provable and $\alpha_1 \simeq \beta_1$ by assumption, $\beta_1, \Gamma_1 \to \Gamma_2$ is also provable, which implies that the desired sequent $\beta_1, \Gamma_1'', \Gamma_2' \to$ is provable, as follows:

$$\frac{\dfrac{\beta_1, \Gamma_1 \to \Gamma_2}{\beta_1, \Gamma_1, \Gamma_2' \to} \quad ('\to)}{\beta_1, \Gamma_1'', \Gamma_2' \to} \quad (''\to)$$

5. The case that the upper sequent of $('\to)$ is obtained as the lower sequent of $(\to')$: We can proceed similarly to (4-j). The case that the last inference of the proof of the sequent $\alpha_1'', \Gamma \to \Delta$ is $('\to')$: This case is divided into several subcases according to how the upper sequent of $('\to')$ is obtained.

(a) The case that the upper sequent of $('\to')$ is an axiom sequent: The treatment of this case is similar to (4-a) and is safely left to the reader.

(b) The case that the upper sequent of $('\to')$ is obtained as the lower sequent of (extension): This case can safely be left to the reader.

(c) The case that the upper sequent of $('\to')$ is obtained as the lower sequent of $(\wedge \to)$: The last two steps of the proof that we must consider go as follows:

$$\dfrac{\dfrac{\alpha, \Delta_1 \to \alpha'_1, \Gamma_1}{\alpha \wedge \beta, \Delta_1 \to \alpha'_1, \Gamma_1} \quad (\wedge \to)}{\alpha''_1, \Gamma'_1 \to (\alpha \wedge \beta)', \Delta'_1} \quad ('\to')$$

The sequent $\alpha''_1, \Gamma'_1 \to \alpha', \Delta'_1$ is provable with a shorter proof than that of $\alpha''_1, \Gamma'_1 \to (\alpha \wedge \beta)', \Delta'_1$ as follows:

$$\dfrac{\alpha, \Delta_1 \to \alpha'_1, \Delta'_1}{\alpha''_1, \Gamma_1 \to \alpha', \Delta'_1} \qquad ('\to')$$

Thus the sequent $\beta_1, \Gamma'_1 \to \alpha', \Delta'_1$ is provable by the induction hypothesis, which implies that the desired sequent $\beta_1, \Gamma'_1 \to (\alpha \wedge \beta)', \Delta'_1$ is also provable as follows:

$$\dfrac{\beta_1, \Gamma'_1 \to \alpha', \Delta'_1}{\beta_1, \Gamma'_1 \to (\alpha \wedge \beta)', \Delta'_1} \qquad (\to \wedge')$$

(d) The case that the upper sequent of $('\to')$ is obtained as the lower sequent of $(\to \vee)$: The treatment is similar to (5-c) and is safely left to the reader.

(e) The case that the upper sequent of $('\to')$ is obtained as the lower sequent of $(\vee \to)$: The last two steps of the proof that we have to consider go as follows:

$$\dfrac{\dfrac{\alpha \to \alpha'_1, \Gamma_1 \qquad \beta \to \alpha'_1, \Gamma_1}{\alpha \vee \beta \to \alpha'_1, \Gamma_1} \quad (\vee \to)}{\alpha''_1, \Gamma'_1 \to (\alpha \vee \beta)'} \quad ('\to')$$

The sequents $\alpha''_1, \Gamma_1 \to \alpha'$ and $\alpha''_1, \Gamma'_1 \to \beta'$ are provable with shorter proofs than that of $\alpha''_1, \Gamma'_1 \to (\alpha \vee \beta)'$ as follows:

$$\dfrac{\alpha \to \alpha'_1, \Gamma_1}{\alpha''_1, \Gamma'_1 \to \alpha'} \quad ('\to') \qquad\qquad \dfrac{\beta \to \alpha'_1, \Gamma_1}{\alpha''_1, \Gamma'_1 \to \beta'} \quad ('\to')$$

Thus the sequents $\beta_1, \Gamma'_1 \to \alpha'$ and $\beta_1, \Gamma'_1 \to \beta'$ are provable by the induction hypothesis, which implies that the desired sequent $\alpha''_1, \Gamma'_1 \to (\alpha \vee \beta)'$ is provable, as follows:

$$\dfrac{\beta_1, \Gamma'_1 \to \alpha' \qquad \beta_1, \Gamma'_1 \to \beta'}{\beta_1, \Gamma'_1 \to (\alpha \vee \beta)'} \qquad (\to \vee')$$

(f) The case that the upper sequent of $('\to')$ is the lower sequent of $(\to')$: The last two steps of the proof that we should consider can be supposed to be one of the following two forms:

$$\cfrac{\cfrac{\alpha_1, \Gamma_1 \to \Gamma_2}{\to \alpha_1', \Gamma_1', \Gamma_2} \quad (\to')}{\alpha_1'', \Gamma_1'', \Gamma_2' \to} \quad ('\to')$$

$$\cfrac{\cfrac{\Gamma_1 \to \alpha_1', \Gamma_2}{\to \alpha_1', \Gamma_1', \Gamma_2} \quad (\to')}{\alpha_1'', \Gamma_1'', \Gamma_2' \to} \quad ('\to')$$

In the former case the sequent $\alpha_1, \Gamma_1'', \Gamma_2' \to$ is provable as follows:

$$\cfrac{\cfrac{\alpha_1, \Gamma_1 \to \Gamma_2}{\alpha_1, \Gamma_1, \Gamma_2' \to} \quad ('\to)}{\alpha_1, \Gamma_1'', \Gamma_2' \to} \quad (''\to)$$

Since $\alpha_1 \simeq \beta_1$ by assumption, the desired sequent $\beta_1, \Gamma_1'', \Gamma_2' \to$ is also provable. As for the latter case, the sequent $\alpha_1'', \Gamma_2' \to \Gamma_1'$ is provable with a shorter proof than that of the sequent $\alpha_1'', \Gamma_1'', \Gamma_2 \to$, as follows:

$$\cfrac{\Gamma_1 \to \alpha_1', \Gamma_2}{\alpha_1'', \Gamma_2' \to \Gamma_1'} \quad ('\to')$$

By the induction hypothesis the sequent $\beta_1, \Gamma_2' \to \Gamma_1'$ is also provable, which implies that the desired sequent $\beta_1, \Gamma_1'', \Gamma_2' \to$ is provable, as follows:

$$\cfrac{\beta_1, \Gamma_2' \to \Gamma_1'}{\beta_1, \Gamma_1'', \Gamma_2' \to} \quad ('\to)$$

(g) The case that the upper sequent of $('\to')$ is obtained as the lower sequent of $(''\to)$ or $(\to'')$: The treatment is similar to (4-c) and is safely left to the reader.

(h) The case that the upper sequent of $('\to')$ is the lower sequent of another $('\to')$: The last two steps of the proof that we have to consider go as follows:

$$\cfrac{\cfrac{\alpha_1, \Gamma_1 \to \Delta_1}{\Delta_1' \to \alpha_1', \Gamma_1'} \quad ('\to')}{\alpha_1'', \Gamma_1'' \to \Delta_1''} \quad ('\to')$$

Since the sequent $\alpha_1, \Gamma_1 \to \Delta_1$ has a shorter proof than the sequent $\alpha_1'', \Gamma_1'' \to \Delta_1''$, the sequent $\beta_1, \Gamma_1 \to \Delta_1$ is also provable by the induction hypothesis, which implies that the desired sequent $\beta_1, \Gamma_1'' \to \Delta_1''$ is also provable, as follows

$$\cfrac{\cfrac{\beta_1, \Gamma_1 \to \Delta_1}{\beta_1, \Gamma_1'' \to \Delta_1} \quad (''\to)}{\beta_1, \Gamma_1'' \to \Delta_1''} \quad (\to'')$$

(i) The case that the upper sequent of $('\to')$ is obtained as the lower sequent of $(\vee' \to), (\to \wedge'), (\wedge' \to),$ or $(\to \vee')$: These four cases can be
. dealt with similarly, so here we deal only with the case of $(\to \vee')$, in which the last two steps of the proof that we must consider go as follows:

$$\frac{\dfrac{\Delta_1 \to \alpha' \qquad \Delta_1 \to \beta'}{\Delta_1 \to (\alpha \vee \beta)'} \quad (\to \vee')}{(\alpha \vee \beta)'' \to \Delta_1'} \quad ('\to')$$

Here α_1 is supposed to be $\alpha \vee \beta$. The sequents $\alpha'' \to \Delta_1'$ and $\beta'' \to \Delta_1'$ are provable with shorter proofs than that of $(\alpha \vee \beta)'' \to \Delta_1'$ as follows:

$$\frac{\Delta_1 \to \alpha'}{\alpha'' \to \Delta_1'} \quad ('\to') \qquad\qquad \frac{\Delta_1' \to \beta''}{\beta'' \to \Delta_1'} \quad ('\to')$$

Therefore the sequents $\alpha \to \Delta_1'$ and $\beta \to \Delta_1'$ are provable by the induction hypothesis, which implies that the sequent $\alpha \vee \beta \to \Delta_1'$ is also provable, as follows:

$$\frac{\alpha \to \Delta_1 \qquad \beta \to \Delta_1}{\alpha \vee \beta \to \Delta_1} \qquad (\vee \to)$$

Since $\beta_1 \simeq \alpha_1 = \alpha \vee \beta$ by assumption, the desired sequent $\beta_1 \to \Delta_1$ is provable.

(j) The case that the upper sequent of $('\to')$ is obtained as the lower sequent of $('\to \wedge)$: The last two steps of the proof that we must consider go as follows:

$$\frac{\dfrac{\alpha' \to \alpha_1', \Gamma_1 \qquad \beta' \to \alpha_1', \Gamma_1}{\to \alpha_1', \Gamma_1', \alpha \wedge \beta} \quad ('\to \wedge)}{\alpha_1'', \Gamma_1', (\alpha \wedge \beta)' \to} \quad ('\to')$$

The sequents $\Gamma_1', \alpha_1'' \to \alpha''$ and $\Gamma_1', \alpha_1'' \to \beta''$ are provable with shorter proofs than that of $\alpha_1', \Gamma_1', (\alpha \wedge \beta)'' \to$ as follows:

$$\frac{\alpha' \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \to \alpha''} \quad ('\to') \qquad\qquad \frac{\beta' \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \to \beta''} \quad ('\to')$$

By the induction hypothesis the sequents $\beta_1, \Gamma_1' \to \alpha$ and $\beta_1, \Gamma_1' \to \beta$ are provable, which implies that the desired sequent $\beta_1, \Gamma_1', (\alpha \wedge \beta)' \to$ is also provable, as follows:

$$\frac{\dfrac{\beta_1, \Gamma_1' \to \alpha \qquad \beta_1, \Gamma_1' \to \beta}{\beta_1, \Gamma_1' \to \alpha \wedge \beta} \quad (\to \wedge)}{\beta_1, \Gamma_1, (\alpha \wedge \beta)' \to} \quad ('\to)$$

■

COROLLARY 11.

1. *If a sequent* $\Gamma, \Pi' \to \Delta, \Sigma'$ *is provable, then the sequent* $\Delta', \Sigma \to \Gamma', \Pi$ *is also provable.*

2. *If a sequent* $\Gamma \to \Delta'$ *is provable, then the sequent* $\Gamma, \Delta \to$ *is provable.*

3. *If a sequent* $\Gamma' \to \Delta$ *is provable, then the sequent* $\to \Gamma, \Delta$ *is also provable.*

Proof. By Theorem 8 it suffices only to take into account the rules $('\to')$, $('\to)$, and $(\to')$. ∎

THEOREM 12. *(The second duality theorem). If* $\alpha_1 \simeq \beta_1$ *and* $\alpha_2 \simeq \beta_2$, *then* $\alpha_1 \wedge \alpha_2 \simeq (\beta_1' \vee \beta_2')'$ *and* $\alpha_1 \vee \alpha_2 \simeq (\beta_1' \wedge \beta_2')'$.

Proof. First we show the following claim:

CLAIM 13.

1. If a sequent $\alpha_1 \wedge \alpha_2, \Gamma \to \Delta$ is provable, then the sequent $(\beta_1' \vee \beta_2')', \Gamma \to \Delta$ is also provable.

2. If a sequent $\Gamma \to \Delta, \alpha_1 \wedge \alpha_2$ is provable, then the sequent $\Gamma \to \Delta, (\beta_1' \vee \beta_2')'$ is also provable.

3. If a sequent $\alpha_1 \vee \alpha_2, \Gamma \to \Delta$ is provable, then the sequent $(\beta_1' \wedge \beta_2'), \Gamma \to \Delta$ is also provable.

4. If a sequent $\Gamma \to \Delta, \alpha_1 \vee \alpha_2$ is provable, then the sequent $\Gamma \to \Delta, (\beta_1' \wedge \beta_2')'$ is also provable.

Here we deal only with the second statement in the above claim, leaving the remaining three statements to the reader. The proof is carried out by induction on the construction of a proof P of the sequent $\Gamma \to \Delta, \alpha_1 \wedge \alpha_2$. Here we deal only with the critical case in which the last inference is $(\to \wedge)$ as follows:

$$\frac{\Gamma \to \alpha_1 \qquad \Gamma \to \alpha_2}{\Gamma \to \alpha_1 \wedge \alpha_2} \qquad (\to \wedge)$$

Since $\alpha_1 \simeq \beta_1$ and $\alpha_2 \simeq \beta_2$ by assumption, the sequents $\Gamma \to \beta_1$ and $\Gamma \to \beta_2$ are provable, which implies that the sequent $\Gamma'' \to (\beta_1' \vee \beta_2')$ is is provable, as follows:

$$\frac{\dfrac{\Gamma \to \beta_1}{\beta_1' \to \Gamma'}('\to') \qquad \dfrac{\Gamma \to \beta_2}{\beta_2' \to \Gamma'} \quad ('\to')}{\dfrac{\beta_1' \vee \beta_2' \to \Gamma'}{\Gamma'' \to (\beta_1' \vee \beta_2')'} \quad ('\to')} \quad (\vee \to)$$

Therefore the sequent $\Gamma \to (\beta_1' \vee \beta_2')'$ s provable by Theorem 8. To establish the remaining half of the theorem smoothly, we introduce a useful notion weaker than provability equivalence. A wff β is said to *be provably dominated by* a wff α, in notation $\alpha \simeq \beta$, if we have that for any finite sets Γ rand Δ of wffs:

1. Whenever the sequent $\alpha, \Gamma \to \delta$ is provable, the sequent $\beta, \Gamma \to \Delta$ is also provable.

2. Whenever the sequent $\Gamma \to \Delta, \alpha$ is provable, the sequent $\Gamma \to \Delta, \beta$ is also provable.

We notice that what we have really proved in Claim 13 is that if $\gamma_1 \widetilde{\to} \delta_1$ and $\gamma_2 \widetilde{\to} \delta_2$, then $\gamma_1 \wedge \gamma_2 \widetilde{\to} (\delta_1' \vee \delta_2')'$ and $\gamma_1 \vee \gamma_2 \widetilde{\to} (\delta_1' \wedge \delta_2')'$. Similarly, what we have really proved in the proof of Theorem 7 is that if $\gamma_1 \widetilde{\to} \delta_1$ and $\gamma_2 \widetilde{\to} \delta_2$, then $\gamma_1' \widetilde{\to} \delta_1'$, $\gamma_1 \wedge \gamma_2 \widetilde{\to} \delta_1 \wedge \delta_2$ nd $\gamma_1 \vee \gamma_2 \widetilde{\to} \delta_1 \vee \delta_2$, while what we have really proved in the proof of Theorem 8 is that if $\alpha \widetilde{\to} \beta$, , then $\alpha'' \widetilde{\to} \beta$. It is easy to see that two wffs α and β are provably equivalent iff each of them is provably dominated by the other. Thus, to conclude the proof of the theorem, it suffices to notice that

$$\alpha_1 \wedge \alpha_2 \widetilde{\to} (\beta_1' \vee \beta_2')' \widetilde{\to} (\alpha_1'' \wedge \alpha_2'') \widetilde{\to} \alpha_1 \wedge \alpha_2$$

■

COROLLARY 14.

1. If $\alpha_1 \simeq \beta_1$ and $\alpha_2 \simeq \beta_2$, then $\alpha_1' \wedge \alpha_2' \simeq (\beta_1 \vee \beta_2)'$ and $\alpha_1' \vee \alpha_2' \simeq (\beta_1 \wedge \beta_2)'$.

Proof. By Theorems 7, 8, and 12, we have that $\alpha_1' \wedge \alpha_2' \simeq (\alpha_1'' \vee \alpha_2'')' \simeq (\beta_1 \vee \beta_2)'$ and $\alpha_1' \vee \alpha_2' \simeq (\alpha_1'' \wedge \alpha_2'')' \simeq (\beta_1 \wedge \beta_2)'$. ■

5   THE CUT-ELIMINATION THEOREM

THEOREM 15.   *A sequent $\alpha, \beta, \Gamma \to \Delta$ is provable iff the sequent $\alpha \wedge \beta, \Gamma \to \Delta$ is provable. Similarly, a sequent $\Pi \to \Sigma, \gamma, \delta$ is provable iff the sequent $\Pi \to \Sigma, \gamma \vee \delta$ is provable.*

Proof. For both statements, the only-if part follows readily from $(\wedge \to)$ or $(\to \vee)$. The if part can be established by induction on the construction of a proof of $\alpha \wedge \beta, \Gamma \to \Delta$ or $\Gamma \to \Delta, \alpha \vee \beta$. ■

COROLLARY 16.   *A sequent $\alpha', \beta', \Gamma \to \Delta$ is provable iff the sequent $(\alpha \vee \beta)', \Gamma \to \Delta$ is provable. Similarly, a sequent $\Pi \to \Sigma, \gamma', \delta'$ is provable iff the sequent $\Pi \to \Sigma, (\gamma \wedge \delta)'$ is provable.*

Proof. Follows from Corollary 1 and Theorem 15. ■

THEOREM 17.   *If a sequent $\alpha \vee \beta, \Gamma \to \Delta$ is provable, then the sequents $\alpha, \Gamma \to \Delta$ and $\beta, \Gamma \to \Delta$ are provable. Similarly, if a sequent $\Pi \to \Sigma, \gamma \wedge \delta$ is provable, then the sequents $\Pi \to \Sigma, \gamma$ and $\Pi \to \Sigma, \delta$ are provable.*

Proof. By induction on the construction of a proof of $\alpha \vee \beta, \Gamma \to \Delta$ or $\Pi \to \Sigma, \gamma \wedge \delta$. Here we deal only with the case that the last step of a proof of a sequent $\alpha \vee \beta, \Gamma \to \Delta$ is $(\vee \to')$. So it must be one of the following two forms.

$$\frac{\Gamma \to \alpha' \quad \Gamma \to \beta'}{\alpha \vee \beta, \Gamma \to} \ (\vee \to')$$

$$\frac{\alpha \vee \beta, \Gamma_1 \to \sigma' \quad \alpha \vee \beta, \Gamma_1 \to \rho'}{\alpha \vee \beta, \sigma \vee \rho, \Gamma_1 \to} \ (\vee \to')$$

In the former case the sequents $\alpha'', \Gamma \to$ and $\beta'', \Gamma \to$ are provable by $('\to)$. So the desired sequents $\alpha, \Gamma \to$ and $\beta, \Gamma \to$ are provable by Theorem 8. In the latter case the sequents $\alpha, \Gamma_1 \to \sigma', \beta, \Gamma_1 \to \sigma'$, and $\beta, \Gamma_1 \to \rho'$, are provable by the induction hypothesis. So the desired sequents $\alpha, \sigma \vee \rho, \Gamma_1 \to$ and $\beta, \sigma \wedge \rho, \Gamma_1 \to$ are provable as follows:

$$\frac{\alpha, \Gamma_1 \to \sigma' \quad \alpha, \Gamma_1 \to \rho'}{\alpha, \sigma \vee \rho, \Gamma_1 \to} \ (\vee \to')$$

$$\frac{\beta, \Gamma_1 \to \sigma' \quad \beta, \Gamma_1 \to \rho'}{\beta, \sigma \vee \rho, \Gamma_1 \to} \ (\vee \to')$$

■

COROLLARY 18. *If a sequent $(\alpha \wedge \beta)', \Gamma \to \Delta$ is provable, then the sequents $\alpha', \Gamma \to \Delta$ and $\beta', \Gamma \to \Delta$ are provable. Similarly, if a sequent $\Pi \to \Sigma, (\gamma \vee \delta)'$ is provable, then the sequents $\Pi \to \Sigma, \gamma'$ and $\Pi \to \Sigma, \delta'$ are provable.*

Proof. This follows from Theorem 17 and Corollary 1. ■

THEOREM 19. *(The cut-elimination theorem). If sequents $\Gamma_1 \to \Delta_1, \alpha$ and $\alpha, \Gamma_2 \to \Delta_2$ are provable with either $\Delta_1 = \varnothing$ or $\Gamma_2 = \varnothing$, then the sequent $\Gamma_1, \Gamma_2 \to \Sigma_1, \Delta_2$ is also provable. In other words, $(cut)_q$ is permissible in **GMQL**.*

Proof. Suppose that the sequents $\Gamma_1 \to \Delta_1, \alpha$ and $\alpha, \Gamma_2 \to \Delta_2$ have proofs P_1 and P_2, respectively. We prove the theorem by double induction principally on $\mathcal{G}(\alpha)$ and secondarily on $l(P_1) + l(P_2)$. By Theorem 12 we can assume that there is no occurrence of the disjunction symbol $\vee$ in P_1 or P_2. As in the proof of Theorem 8, whenever we are forced to deal with the rules $(\wedge \to)$ or $(\to \wedge')$, each of which consists of two forms, only one of them is treated. Our proof is divided into several cases according to which inference rule is used in the last step of P_1 or P_2 as follows:

1. The case that one of the sequents $\Gamma_1 \to \Delta_1, \alpha$ and $\alpha, \Gamma_2 \to \Delta_2$ is an axiom sequent: There is nothing to prove.

2. The case that one of the sequents $\Gamma_1 \to \Delta_1, \alpha$ and $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of (extension): Here we deal only with the case that the former sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of (extension), leaving the dual case to the reader. Then the last step of the proof P_1 is in one of the following two forms:

$$\frac{\Gamma_{11} \to \Delta_{11}, \alpha}{\Gamma_{11}, \Gamma_{12} \to \Delta_{11}, \Delta_{12}, \alpha} \quad \text{(extension)}$$

$$\frac{\Gamma_{11} \to \Delta_{11}}{\Gamma_{11}, \Gamma_{12} \to \Delta_{11}, \Delta_{12}, \alpha} \quad \text{(extension)}$$

In the former case the desired sequent $\Gamma_{11}, \Gamma_{12}, \Gamma_2 \to \Delta_{11}, \Delta_{12}, \Delta_2$ is provable by induction hypothesis as follows.

$$\frac{\dfrac{\Gamma_{11} \to \Delta_{11}, \alpha \quad\quad \alpha, \Gamma_2 \to \Delta_2}{\Gamma_{11}, \Gamma_2 \to \Delta_{11}, \Delta_2} \;\; \text{(cut)}_q}{\Gamma_{11}, \Gamma_{12}, \Gamma_2 \to \Delta_{11}, \Delta_{12}, \Delta_2} \quad \text{(extension)}$$

In the latter case the desired sequent $\Gamma_{11}, \Gamma_{12}, \Gamma_2 \to \Delta_{11}, \Delta_{12}, \Delta_2$ is obtained as follows.

$$\frac{\Gamma_{11} \to \Delta_{11}}{\Gamma_{11}, \Gamma_{12}, \Gamma_2 \to \Delta_{11}, \Delta_{12}, \Delta_2} \quad\quad \text{(extension)}$$

3. The case that either the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of one of the inference rules $('' \to)$ and $(\wedge \to)$ or the sequent $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of one of the inference rules $(\to '')$ and $(\to \wedge')$: Here we deal only with the case that the sequent $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of $(\to \wedge')$, leaving the remaining three cases to the reader. So the last step of P_2 is of the following form:

$$\frac{\alpha, \Gamma_2 \to \Sigma_2, \beta'}{\alpha, \Gamma_2 \to \Sigma_2, (\beta \vee \gamma)'} \quad\quad (\to \wedge')$$

The desired sequent $\Gamma_1, \Gamma_2 \to \Delta_1, \Sigma_2, (\beta \wedge \gamma)'$ is provable by induction hypothesis as follows:

$$\frac{\dfrac{\Gamma_1 \to \Delta_1, \alpha \quad\quad \alpha, \Gamma_2 \to \Sigma_2, \beta'}{\Gamma_1, \Gamma_2 \to \Delta_1, \Sigma_2, \beta'} \;\; \text{(cut)}_q}{\Gamma_1, \Gamma_2 \to \Delta_1, \Sigma_2, (\beta \wedge \gamma)'} \quad (\to \wedge')$$

4. The case that either the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of $(\wedge' \to)$ or the sequent $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of $(\to \wedge)$: Here we deal only with the former case, leaving a similar treatment of the latter case to the reader. So the last step of P_1 goes as follows:

$$\frac{\beta' \to \Delta_1, \alpha \qquad \gamma' \to \Delta_1, \alpha}{(\beta \wedge \gamma)' \to \Delta_1, \alpha} \qquad (\wedge' \to)$$

If $\gamma_2 = \varnothing$, then the desired sequent $(\beta \wedge \gamma)' \to \Delta_1, \Delta_2$ is provable by the induction hypothesis as follows:

$$\frac{\dfrac{\beta' \to \Delta_1, \alpha \quad \alpha \to \Delta_2}{\beta_1' \to \Delta_1, \Delta_2} \ (\text{cut})_q \qquad \dfrac{\gamma' \to \Delta_1, \alpha \quad \alpha \to \Delta_2}{\gamma_2' \to \Delta_1, \Delta_2} \ (\text{cut})_q}{(\beta \wedge \gamma)' \to \Delta_1, \Delta_2} \ (\wedge' \to)$$

Unless $\Gamma_2 = \varnothing$, the situation can be classified into cases according to which inference rule is used in the last step of P_2. If $\Gamma_2 \neq \varnothing$ and it is not the case that the last inference of P_2 is $(\to \wedge)$, the situation is subsumed under the cases that have been or will be dealt with. If $\Gamma_2 \neq \varnothing$ and the last inference of P_2 is $(\to \wedge)$, then surely $\Gamma_1 \neq \varnothing$, so that the situation can be handled dually to the case that $\Gamma_2 = \varnothing$.

5. The case that either the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of one of the inference rules $(\to'')$ and $(\to \wedge')$ or the sequent $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of one of the inference rules $(''\to)$ and $(\wedge \to)$: Here we deal only with the case that the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of $(\to \wedge')$, leaving the remaining three cases to the reader. So the last step of P_1 is in one of the following two forms:

$$\frac{\Gamma_1 \to \Sigma, \beta', \alpha}{\Gamma_1 \to \Sigma, (\beta \wedge \gamma)', \alpha} \qquad (\to \wedge')$$

$$\frac{\Gamma_1 \to \Delta_1, \beta'}{\Gamma_1 \to \Delta_1, (\beta \wedge \gamma)'} \qquad (\to \wedge')$$

In the latter case α is supposed to be $(\beta \wedge \gamma)'$. In the former case the $(\text{cut})_q$ at issue is an instance of (cut-1), so that $\Gamma_2 = \varnothing$, and the desired sequent $\Gamma_1 \to \Sigma, (\beta \wedge \gamma)', \Delta_2$ is provable by induction hypothesis, as follows:

$$\frac{\dfrac{\Gamma_1 \to \Sigma, \beta', \alpha \quad \alpha \to \Delta_2}{\Gamma_1 \to \Sigma, \beta'} \ (\text{cut})_q}{\Gamma_1 \to \Sigma, (\beta \wedge \gamma)'} \ (\to \wedge')$$

As for the latter case, the cut formula is $(\beta \wedge \gamma)'$, and the sequent, $\beta', \Gamma_2 \to \Delta_2$ is provable by Corollary 18. Thus the desired sequent $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ is provable by the induction hypothesis, as follows:

$$\frac{\Gamma_1 \to \Delta_1, \beta' \qquad \beta', \Gamma_2 \to \Delta_2}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2} \ (\text{cut})_q$$

6. The case that either the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of $(\to \wedge)$ or the sequent $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent . of $(\wedge' \to)$: Here we deal only with the latter case, leaving a similar treatment of the latter to the reader. So the last step of P_2 goes as follows:

$$\frac{\beta' \to \Delta_2 \qquad \gamma' \to \Delta_2}{(\beta \wedge \gamma)' \to \Delta_2} \quad (\wedge' \to)$$

Here α is supposed to be $(\beta \wedge \gamma)'$, and the $(\mathrm{cut})_q$ at issue is an instance of (cut-1) with the cut formula $(\beta \wedge \gamma)'$. By Corollary 8 the sequent $\Gamma_1 \to \Delta_1, \beta', \gamma'$ is provable, so that the desired sequent $\Gamma_1 \to \Delta_1, \Delta_2$ is also provable, as follows:

$$\frac{\dfrac{\Gamma_1 \to \Delta_1, \beta', \gamma' \qquad \beta' \to \Delta_2}{\Gamma_1 \to \Delta_1, \Delta_2, \gamma' \qquad \gamma' \to \Delta_2} \quad (\mathrm{cut})_q}{\Gamma_1 \to \Delta_1, \Delta_2} \quad (\mathrm{cut})_q$$

7. The case that the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of $('\to \wedge)$: The last step of P_1 is in one of the following two forms:

$$\frac{\beta' \to \Sigma, \alpha \qquad \gamma' \to \Sigma, \alpha}{\to \Sigma, \beta \wedge \gamma, \alpha} \quad ('\to \wedge)$$

$$\frac{\beta' \to \Delta_1 \qquad \gamma' \to \Delta_1}{\to \Delta_1, \beta \wedge \gamma} \quad ('\to \wedge)$$

In the latter case α is assumed to be $\beta \wedge \gamma$. First we deal with the former case, in which the $(\mathrm{cut})_q$ at issue is (cut-1) so that $\Gamma_2 = \varnothing$. Then the desired sequent $\to \Sigma, \beta \wedge \gamma, \Delta_2$ is provable by the induction hypothesis as follows:

$$\frac{\dfrac{\beta' \to \Sigma, \alpha \qquad \alpha \to \Delta_2}{\beta_1' \to \Sigma, \Delta_2} \ (\mathrm{cut})_q \qquad \dfrac{\gamma' \to \Sigma, \alpha \qquad \alpha \to \Delta_2}{\gamma' \to \Sigma, \Delta_2} \ (\mathrm{cut})_q}{\to \Sigma, \beta \wedge \gamma, \Delta_2} \quad ('\to \wedge)$$

As for the latter case, suppose first that $\Delta_1 \neq \varnothing$, so that $\Gamma_2 = \varnothing$. Then the sequents $\Delta_1' \to \beta$ and $\Delta_1' \to \gamma$ are provable by Corollary 11, while the sequent $\beta, \gamma \to \Delta_1'', \Delta_2$ is provable by Theorem 15. Thus the sequent $\to \Delta_1'', \Delta_2$ is provable by the induction hypothesis, as follows:

$$\frac{\dfrac{\dfrac{\Delta_1' \to \beta \qquad \beta, \gamma \to \Delta_2}{\Delta_1' \to \gamma \qquad \gamma, \Delta_1' \to \Delta_2} \ (\mathrm{cut})_q}{\Delta_1' \to \Delta_2} \ (\mathrm{cut})_q}{\to \Delta_1'', \Delta_2} \quad (\to')$$

Thus the desired sequent $\to \Delta_1, \Delta_2$ is provable by Theorem 8. If $\Delta_1 = \varnothing$, then the sequents $\to \beta$ and $\to \gamma$ are provable by Corollary 11, while the

sequent $\beta, \gamma, \Gamma_2 \to \Delta_2$ is provable by Theorem 15. Thus the desired sequent $\Gamma_2 \to \Delta_2$ is provable by the induction hypothesis as follows:

$$\dfrac{\dfrac{\to \beta \qquad \beta, \gamma, \Gamma_2 \to \Delta_2}{\to \gamma \qquad \gamma, \Gamma_2 \to \Delta_2} \quad (\mathrm{cut})_q}{\Gamma_2 \to \Delta_2} \qquad (\mathrm{cut})_q$$

8. The case that one of the sequents $\Gamma_1 \to \Delta_1, \alpha$ and $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of $(\to')$: Here we deal only with the case that the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of $(\to')$, leaving the dual case to the reader. So the last step of the proof P_1 is in one of the following two forms:

$$\dfrac{\Delta'_{12} \to \Delta_{11}, \alpha}{\to \Delta_{11}, \Delta'_{12}, \alpha} \qquad (\to')$$

$$\dfrac{\Delta'_{12}, \beta \to \Delta_{11}}{\to \Delta_{11}, \Delta'_{12}, \beta'} \qquad (\to')$$

In the latter case α is supposed to be β'. First we deal with the former case. If $\Gamma_2 = \varnothing$, then the desired sequent $\to \Delta_{11}, \Delta'_{12}, \Delta_2$ is provable by the induction hypothesis, as follows:

$$\dfrac{\dfrac{\Delta_{12} \to \Delta_{11}, \alpha \qquad \alpha \to \Delta_2}{\Delta_{12} \to \Delta_{11}, \Delta_2} \quad (\mathrm{cut})_q}{\to \Delta_{11}, \Delta'_{12}, \Delta_2} \quad (\to')$$

If $\Gamma_2 \neq \varnothing$, then α is of the form γ' and the sequent $\Delta_{12} \to \Delta_{11}, \alpha$ is $\gamma \to \gamma'$. The sequents $\gamma \to$ and $\Delta'_2 \to \gamma$ are provable by Corollary 11, which implies that the sequent $\Delta'_2 \to$ is also provable by the induction hypothesis, as follows:

$$\dfrac{\Delta'_2 \to \gamma \qquad \gamma \to}{\Delta'_2 \to} \qquad (\mathrm{cut})_q$$

By Corollary 11 the sequent $\to \Delta_2$ is provable, which implies that the desired sequent $\to \Delta_1, \Delta_2$ is provable as follows:

$$\dfrac{\to \Delta_2}{\to \Delta_1, \Delta_2} \quad (\text{extension})$$

Now we deal with the latter case. If $\Gamma_2 = \varnothing$, then the sequent $\Delta'_2 \to \beta$ is provable by Corollary 11, and the sequent $\to \Delta_{11}, \Delta'_{12}, \Delta''_2$ is also provable by the induction hypothesis as follows: .

$$\dfrac{\dfrac{\Delta'_2 \to \beta \qquad \Delta_{12}, \beta \to \Delta_{11}}{\Delta_{12}, \Delta'_2 \to \Delta_{11}} \quad (\mathrm{cut})_q}{\to \Delta_{11}, \Delta'_{12}, \Delta''_2} \quad (\to')$$

Thus the desired sequent $\rightarrow \Delta_{11}, \Delta_{12}, \Delta_2$ is provable by Theorem 8. If $\Gamma_2 \neq \varnothing$, then the sequent $\Delta_{12}, \beta \rightarrow \Delta_{11}$ must be $\beta \rightarrow$ or $\beta \rightarrow \beta'$, the latter of which implies by Corollary 11 that the sequent $\beta \rightarrow$ is provable. Thus in any case the sequent $\beta \rightarrow$ is provable. Since the sequent $\Delta_2' \rightarrow \Gamma_2', \beta$ is provable by Corollary 11, the sequent $\Delta_2' \rightarrow \Gamma_2'$ is provable by induction hypothesis as follows:

$$\frac{\Delta_2' \rightarrow \Gamma_2', \beta \qquad \beta \rightarrow}{\Delta_2' \rightarrow \Gamma_2'} \qquad (\text{cut})_q$$

Therefore the sequent $\Gamma_2 \rightarrow \Delta_2$ is provable by Corollary 11, which implies that the desired sequent $\Gamma_2 \rightarrow \Delta_1, \Delta_2$ is provable as follows:

$$\frac{\Gamma_2 \rightarrow \Delta_2}{\Gamma_2 \rightarrow \Delta_1, \Delta_2} \quad (extension)$$

9. The case that both the sequent $\Gamma_1 \rightarrow \Delta_1, \alpha$ and the sequent $\alpha, \Gamma_2 \rightarrow \Delta_2$ are obtained as the lower sequent of $('\rightarrow')$: The last steps of the proofs P_1 and P_2 go as follows:

$$\frac{\Sigma_1, \beta \rightarrow \Pi_1}{\Pi_1' \rightarrow \Sigma_1', \beta} \qquad ('\rightarrow')$$

$$\frac{\Sigma_2, \rightarrow \beta, \Pi_2}{\beta', \Pi_2' \rightarrow \Sigma_2'} \qquad ('\rightarrow')$$

In the above α is supposed to be β'. The desired sequent $\Pi_1, \Pi_2 \rightarrow \Sigma_1, \Sigma_2$ is provable by the induction hypothesis as follows:

$$\frac{\dfrac{\Sigma_2 \rightarrow \Pi_2, \beta \qquad \beta, \Sigma_1 \rightarrow \Pi_1}{\Sigma_1, \Sigma_2 \rightarrow \Pi_1, \Pi_2}}{\Pi_1', \Pi_2' \rightarrow \Sigma_1', \Sigma_2'} \quad \begin{array}{l} (cut) \\ ('\rightarrow') \end{array}$$

∎

6   THE COMPLETENESS THEOREM

An *O-frame* is a pair $(X, \perp)$ of a nonempty set X and an orthogonality relation (i.e., an irreflexive and symmetric binary relation) on X. Given $Y \subseteq X$, we write $Y^\perp$ for the set $\{x \in X | x \perp y$ for any $y \in Y\}$. A subset Y of X is said to be $\perp$-*closed* if $Y = Y^{\perp\perp}$.

An *O-model* is a triple $(X, \perp, D)$, where $(X, \perp)$ is an *O-frame* and D assigns to each propositional variable p a $\perp$-closed subset $D(p)$ of X. The notation $\|\alpha\|$ for a wff α is defined inductively as follows:

1. $\|p\| = D(p)$ for any propositional variable p.

2. $\|\alpha \wedge \beta\| = \|\alpha\| \cap \|\beta\|$.

3. $\|\alpha'\| = \|\alpha\|^\perp$.

4. $\|\alpha \vee \beta\| = \|\alpha' \wedge \beta'\| = (\|\alpha\|^\perp \cap \|\beta\|^\perp)^\perp$.

Given $x \in X$ and a wff α, we write $V(\alpha; x) = 1$ if $x \in \|\alpha\|$ and $V(\alpha; x) = 0$ if $x \notin \|\alpha\|$. Given $x \in X$ and a sequent $\Gamma \to \Delta$, we write $V(\Gamma \to \Delta; x) = 1$ if $x \in \bigcap\{\|\alpha\| \mid \alpha \in \Gamma\}$ and $x \notin (\cup\{\|\beta\|^\perp \mid \beta \in \Delta\})^\perp$, and $V(\Gamma \to \Delta; x) = 0$ otherwise.

A sequent $\Gamma \to \Delta$ is said to be *realizable* if there exists an *O-model* $(X, \perp, D)$ and some $x \in X$ such that $V(\Gamma \to \Delta; x) = 1$. The sequent $\Gamma \to \Delta$ is called *valid* otherwise.

THEOREM 20. *(The soundness theorem). If a sequent $\Gamma \to \Delta$ is provable, then it is valid.*

Proof. By induction on the construction of a proof of the sequent $\Gamma \to \Delta$. ∎

A set Ω of wffs is said to be *admissible* if it satisfies the following conditions:

1. If p is a propositional variable and $p \in \Omega$, then $p' \in \Omega$.

2. If $\alpha \in \Omega$ and β is a subformula of α, then $\beta \in \Omega$.

3. If $(\alpha \vee \beta) \in \Omega$, then $(\alpha' \wedge \beta')' \in \Omega$.

A finite set Γ of wffs is said to be *inconsistent* if for some wff α, both of the sequents $\Gamma \to \alpha$ and $\Gamma \to \alpha'$ are provable. Otherwise the set Γ is said to be *consistent*.

LEMMA 21. *A finite set Γ of wffs is inconsistent iff the sequent $\Gamma \to$ is provable.*

Proof. The if part is obvious. The only-if part can be shown easily as follows:

$$\cfrac{\cfrac{\Gamma \to \alpha \quad \Gamma \to \alpha'}{\Gamma \to \alpha \wedge \alpha'} \ (\to \wedge) \qquad \cfrac{\cfrac{\cfrac{\alpha \to \alpha}{\alpha, \alpha' \to} \ ('\to)}{\alpha \wedge \alpha' \to} \ (\wedge \to)}{\qquad}}{\Gamma \to} \ (\text{cut})_q$$

∎

Given an admissible set Ω of wffs, the Ω-*canonical* O-model $\mathcal{M}(\Omega) = (X_\Omega, \perp_\Omega, D_\Omega)$ is defined as follows:

1. X_Ω is the set of all the consistent subsets of Ω.

2. For any $\Gamma_1, \Gamma_2 \in X_\Omega, \Gamma_1 \perp_\Omega \Gamma_2$ iff for some $\alpha' \in \Omega$, either: (a) both of the sequents $\Gamma_1 \to \alpha$ and $\Gamma_1 \to \alpha'$ are provable, or (b) both of the sequents $\Gamma_1 \to \alpha'$ and $\Gamma_2 \to \alpha$ are provable.

3. If $p \notin \Omega$, then $D_\Omega(p) = \varnothing$, while if $p \in \Omega$, then $D_\Omega(p)$ consists of all the consistent subsets Γ of Ω such that the sequent $\Gamma \to p$ is provable.

THEOREM 22. $\mathcal{M}(\Omega)$ *is an O-model.*

Proof. Obviously the relation $\perp_\Omega$ is symmetric. That the relation $\perp_\Omega$ is irreflexive follows from our assumption that every element of X_Ω is a consistent set of wffs. Now it remains to show that $D_\Omega(p)$ is $\perp_\Omega$-closed for any propositional variable p. Unless $p \in \Omega$, there is nothing to prove. So let $p \in \Omega$. Let Γ be an element of X_Ω such that the sequent $\Gamma \to p$ is not provable. Suppose for the sake of contradiction that the set $\{p'\}$ is inconsistent, which implies by Lemma 21 that the sequent $p' \to$ is provable. By Corollary 11 the sequent $\to p$ is provable, which implies by (extension) that the sequent $\Gamma \to p$ is provable. This is a contradiction. So $\{p'\} \in X_\Omega$. Suppose, for the sake of contradiction, that for some $\alpha' \in \Omega$, either both of the sequents $\Gamma \to \alpha'$ and $p' \to \alpha$ are provable or both of the sequents $\Gamma \to \alpha'$ and $p' \to \alpha$ are provable. Here we deal only with the former case, leaving a similar treatment of the latter to the reader. By Corollary 11 the sequent $\alpha \to p$ is provable, which implies by (cut) that the sequent $\Gamma \to p$ is provable. This is a contradiction. Thus it cannot be the case that $\Gamma \perp_\Omega \{p'\}$, while for any $\Delta \in X_\Omega$ such that the sequent $\Delta \to p$ is provable, $\Delta \perp_\Omega \{p\}$. This implies that the set of all $\Delta \in X_\Omega$ such that the sequent $\Delta \to p$ is provable is $\perp_\Omega$-closed. ■

The *disjunction grade* of a wff α, denoted by $\mathcal{G}_\vee(\alpha)$, is defined inductively as follows:

1. $\mathcal{G}_\vee(p) = o$ for any propositional variable p.

2. $\mathcal{G}_\vee(\alpha') = \mathcal{G}_\vee(\alpha)$.

3. $\mathcal{G}_\vee(\alpha \wedge \beta) = \mathcal{G}_\vee(\alpha) + \mathcal{G}_\vee(\beta)$.

4. $\mathcal{G}_\vee(\alpha \vee \beta) = \mathcal{G}_\vee(\alpha) + \mathcal{G}_\vee(\beta) + 1$.

THEOREM 23. *(The fundamental theorem for $\mathcal{M}(\Omega)$). For any $\alpha \in \Omega$ and any $\Gamma \in X_\Omega$, the sequent $\Gamma \to \alpha$ is provable iff $\Gamma \in \|\alpha\|$ in $\mathcal{M}(\Omega)$.*

Proof. The proof is carried out by double induction principally on $\mathcal{G}_\vee(\alpha)$ and secondarily on $\mathcal{G}(\alpha)$. The proof is divided into several cases.

1. In the case that α is a propositional variable: It follows from the definition of D_Ω.

2. In the case that $\alpha = \beta'$ for some wff β: If $\Gamma \to \beta'$ is provable, then $\Gamma \perp_\Omega \|\beta\|$ by induction hypothesis, which implies that $\Gamma \in \|\beta'\|$. Suppose, for the sake of contradiction, that the set $\{\beta\}$ is inconsistent, which implies by Lemma 21 that the sequent $\beta \to$ is provable. Thus the sequent $\Gamma \to \beta'$ is provable as follows:

$$\frac{\dfrac{\beta \to}{\to \beta'} \quad (\to')}{\Gamma \to \beta'} \quad \text{(extension)}$$

This is a contradiction. So it must be the case that $\{\beta\} \in X_\Omega$. Suppose, for the sake of contradiction, that for some $\gamma' \in \Omega$, either both of the sequents $\Gamma \to \gamma'$ and $\beta \to \gamma$ are provable or both of the sequents $\Gamma \to \gamma$ and $\beta \to \gamma'$ are provable. Here we deal only with the former case, leaving safely a similar treatment of the latter to the reader. The desired contradiction is obtained as follows:

$$\frac{\Gamma \to \gamma' \qquad \dfrac{\beta \to \gamma}{\gamma' \to \beta'} \quad ('\to')}{\Gamma \to \beta} \quad \text{(cut)}_q$$

Thus it cannot be the case that $\Gamma \perp_\Omega \{\beta\}$. Since $\{\beta\} \in \|\beta\|$ by induction hypothesis, this means that $\Gamma \notin \|\beta\| \perp \Omega = \|\beta'\|$.

3. In the case that α is of the form $\beta \wedge \gamma$ for some wffs β and γ: If the sequent $\Gamma \to \alpha$ is provable, then both of the sequents $\Gamma \to \beta$ and $\Gamma \to \gamma$ are provable by Theorem 17, which implies by induction hypothesis that $\Gamma \in \|\beta\|$ and $\Gamma \in \|\gamma\|$. So $\Gamma \in \|\beta\| \cap \|\gamma\| = \|\beta \wedge \gamma\|$. Unless the sequent $\Gamma \to \alpha$ is provable, suppose, for the sake of contradiction, that both of the sequents $\Gamma \to \beta$ and $\Gamma \to \gamma$ are provable. The desired conclusion is obtained as follows:

$$\frac{\Gamma \to \beta \qquad \Gamma \to \gamma}{\Gamma \to \beta \wedge \gamma} \qquad (\to \wedge)$$

Thus one of the sequents $\Gamma \to \beta$ and $\Gamma \to \gamma$ is consistent, which implies by induction hypothesis that $\Gamma \notin \|\beta\|$ or $\Gamma \notin \|\gamma\|$. So $\Gamma \notin \|\beta \wedge \gamma\| = \|\beta\| \cap \|\gamma\|$.

4. In the case that α is of the form $\beta \vee \gamma$ for some wffs β and γ: Use Theorem 12.

$\blacksquare$

THEOREM 24. *(The completeness theorem). A sequent $\Gamma \to \Delta$ is realizable iff it is consistent.*

Proof. The only-if part is the soundness theorem already established. To see the if part, take an admissible set Ω such that $\Gamma \cup \{\beta_1 \vee ... \vee \beta_n\} \subseteq \Omega$, where $\Delta = \{\beta_1, ..., \beta_n\}$. By Theorem 15 the sequent $\Gamma \to \Delta$ is consistent iff the sequent $\Gamma \to \beta_1 \vee ... \vee \beta_n$ is consistent. The desired conclusion follows readily from Theorem 23. $\blacksquare$

We remark in passing that in the proof of Theorem 24 it does not matter how to insert parentheses in $\beta_1 \vee ... \vee \beta_n$.

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