A NOTE ON NORMALLY GENERATED LINE BUNDLES ON COMPACT RIEMANN SURFACES

By
Tatsuya Arakawa

1. Introduction

Let $X$ denote a compact Riemann surface of genus $g(X) > 0$ and $L$ an ample line bundle on $X$.

**Definition 1.** $L$ is said to be normally generated if, for each $n > 0$, the natural map

$$\text{Sym}^n H^0(X, L) \rightarrow H^0(X, L^n)$$

is surjective.

There are the following two sufficient conditions for line bundles on $X$ to be normally generated obtained by H. H. Martens and D. Mumford, respectively:

**Theorem 1** (cf. [3]). The canonical bundle $K_X$ on $X$ is normally generated if and only if $X$ is nonhyperelliptic.

**Theorem 2** ([4]). If $\deg L \geq 2g(X) + 1$, then $L$ is normally generated.

On the other hand, Homma [2] classified all the normally generated line bundles on $X$ when the genus of $X$ is three.

**Theorem 3** ([2]). Suppose $g(X) = 3$.

(i) If $X$ is hyperelliptic, then $L$ is normally generated if and only if $\deg L \geq 7$.

(ii) If $X$ is nonhyperelliptic, then $L$ is normally generated if and only if $L$ satisfies one of the following conditions:

(a) $\deg L \geq 7$.

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(b) $\deg L = 6$ and $H^0(X, L \otimes K_X^{-1}) = 0$.

(c) $L \cong K_X$.

Now let $\pi : X \to Y$ be a (possibly ramified) covering of compact Riemann surfaces and let $g(Y) \geq 0$ denote the genus of $Y$.

**Problem.** Classify ample line bundles on $Y$ such that the pull backs on $X$ are normally generated.

In this note, we will study this problem in the cases of $\pi$ being double coverings with small $g(X)$ or $g(Y)$. In §2, we will determine such line bundles on $Y$ when $g(X) = 3$ and in §3, the cases of $Y$ being rational or elliptic Riemann surfaces are treated.

Before closing this section, let us recall some fundamental facts on double coverings (cf. [5]):

**Lemma 1.** Let $B$ denote the branch locus of the double covering $\pi : X \to Y$ on $Y$. Then there exists a line bundle $F$ on $Y$ with $2F \cong B$ such that the following conditions hold:

(i) $X$ is embedded into $F$ and the projection of $F$ to $Y$ restricted on $X$ coincides with $\pi$.

(ii) The canonical bundle $K_X$ on $X$ is linearly equivalent to $\pi^*(K_Y \otimes F)$ where $K_Y$ is the canonical bundle on $Y$.

(iii) For any line bundle $L$ on $Y$, we have:

$$\pi_*\mathcal{O}_Y(\pi^*L) \cong \mathcal{O}_Y(L) \oplus \mathcal{O}_Y(L \otimes F^{-1}).$$

**Corollary.** Let $\pi : X \to Y$ be a double covering of compact Riemann surfaces. Then the induced homomorphism $\pi^* : \text{Pic } Y \to \text{Pic } X$ is injective.

**Proof.** Let $M$ be a line bundle on $Y$ such that the pull back $\pi^*M$ is trivial on $X$. Then we have $\deg M = 0$ and $h^0(X, \pi^*M) = 1$. Hence, by Lemma 1 (iii), we have $h^0(Y, M) = h^0(X, \pi^*M) - h^0(Y, M \otimes F^{-1}) = 1$, that is, $M$ is also trivial on $Y$.

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2. The cases of \(g(X) = 3\)

By Lemma 1 and Theorem 3, we can determine such line bundle \(M\) on \(Y\) as in Problem when \(g(X) = 3\):

Since \(g(X) > g(Y)\), we have \(g(Y) = 0, 1\) or \(2\). If \(g(Y) = 0\), then \(X\) is hyperelliptic. On the other hand, we have the following result of Farkas:

**Lemma 2 ([1]).** Let \(X \to Y\) be a double covering of compact Riemann surfaces with \(g(X) = 3\) and \(g(Y) = 2\). Then \(X\) is hyperelliptic.

As a conclusion of Lemma 2 and Theorem 3 (i), we obtain the following result.

**Proposition 1.** Suppose \(g(Y) = 0\) or \(2\). Then \(\pi^* M\) is normally generated if and only if \(\deg M \geq 4\).

Now suppose \(X\) is nonhyperelliptic. Then we have \(g(Y) = 1\) and hence, by Lemma 1 (ii), \(K_X \cong \pi^* F\) and \(\deg F = 2\).

By Theorem 3 (ii), \(\pi^* M\) is normally generated if \(\deg M \geq 4\) and not normally generated if \(\deg M = 1\).

Suppose \(\deg M = 2\). Then \(\pi^* M\) is normally generated if and only if \(\pi^* M \cong \pi^* F\), that is, by the corollary to Lemma 1, \(M \cong F\).

Suppose \(\deg M = 3\). Then, since \(g(Y) = 1\) and \(\deg M \otimes F^{-1} = 1\), we have

\[
h^0(X, \pi^* M \otimes K_X^{-1}) = h^0(Y, M \otimes F^{-1}) + h^0(Y, M \otimes F^{-2}) > 0.
\]

Consequently we get the following:

**Proposition 2.** Suppose \(g(Y) = 1\).

(i) If \(X\) is hyperelliptic then \(\pi^* M\) is normally generated if and only if \(\deg M \geq 4\).

(ii) If \(X\) is nonhyperelliptic then \(\pi^* M\) is normally generated if and only if \(\deg M \geq 4\) or \(M \cong F\).

3. The cases of \(g(X) \geq 4\) and \(g(Y) \leq 1\)

3.1. The cases of \(g(Y) = 0\)

If \(g(Y) = 0\), then \(\deg F = g(X) + 1\) and hence, if \(\deg M < g(X) + 1\) for a line bundle \(M\) on \(Y\), we have

\[
H^0(X, \pi^* M) \cong H^0(Y, M)
\]
by Lemma 1 (iii), that is, each section in $H^0(X, \pi^*M)$ is the pull back of a section in $H^0(Y, M)$. But, for a sufficiently large $n$,

$$H^0(X, \pi^*M^n) \neq H^0(Y, M^n)$$

by Lemma 1 (iii) again. We therefore conclude that $\pi^*M$ is not normally generated in this case.

On the other hand, by Theorem 2, $\pi^*M$ is normally generated if $\deg M \geq g(X) + 1$.

Consequently we have:

**Proposition 3.** Suppose $g(Y) = 0$. Then $\pi^*M$ is normally generated if and only if $\deg M \geq g(X) + 1$.

3.2. The cases of $g(Y) = 1$

If $g(Y) = 1$, then we have $\deg F = g(X) - 1$ and $K_X \equiv \pi^*F$. If moreover $g(X) \geq 4$, then $X$ is always nonhyperelliptic.

Now by the same arguments as in §3.1, we can conclude that, for a line bundle $M$ on $Y$, $\pi^*M$ is not normally generated if $H^0(X, M \otimes F^{-1}) = 0$. Therefore if $\pi^*M$ is normally generated, then $\deg M \geq g(X)$ or $M \equiv F$. On the other hand, by Theorems 1 and 2, $\pi^*M$ is normally generated if $\deg M \geq g(X) + 1$ or $M \equiv F$.

Now suppose $\deg M = g(X)$. By Lemma 1 (iii), we have

$$H^0(X, \pi^*M) \cong H^0(Y, M) \oplus H^0(Y, M \otimes F^{-1})$$

and

$$H^0(X, \pi^*M^2) \cong H^0(Y, M^2) \oplus H^0(Y, M^2 \otimes F^{-1}).$$

But by the Riemann-Roch theorem, we have $h^0(Y, M) = g$, $h^0(Y, M \otimes F^{-1}) = 1$ and $h^0(Y, M^2 \otimes F^{-1}) = g + 1$. Hence the natural map

$$H^0(Y, M) \otimes H^0(Y, M \otimes F^{-1}) \rightarrow H^0(Y, M^2 \otimes F^{-1})$$

is not surjective, and neither is

$$H^0(X, \pi^*M) \otimes H^0(X, \pi^*M) \rightarrow H^0(X, \pi^*M^2).$$

Therefore we conclude that, in this case, $\pi^*M$ is not normally generated.
Consequently we have:

**Proposition 4.** Suppose \( g(X) \geq 4 \) and \( g(Y) = 1 \). Then, for a line bundle \( M \) on \( Y \), \( \pi^*M \) is normally generated if and only if \( \deg M \geq g(X) + 1 \) or \( M \cong F \).

**References**


Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka Osaka 560-0043
Japan

*E-mail address: arakawa@math.sci.osaka-u.ac.jp*