HELICES AND ISOMETRIC IMMERSIONS

By

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Abstract. Let \( f : M \to \tilde{M} \) be an isometric immersion of a Riemannian manifold \( M \) into a Riemannian manifold \( \tilde{M} \). We study the geometry of submanifolds under various assumptions with respect to the first curvature \( \tilde{\lambda}_1 \) and the second curvature \( \tilde{\lambda}_2 \) of \( \tilde{\sigma} = f \circ \sigma \) in \( \tilde{M} \) for a helix \( \sigma \) in \( M \).

Introduction

Let \( f : M \to \tilde{M} \) be an isometric immersion of a Riemannian manifold \( M \) into a Riemannian manifold \( \tilde{M} \). K. Nomizu and K. Yano [4] proved the following fact:

If, for some \( r > 0 \), every circle of radius \( r \) in \( M \) is a circle in \( \tilde{M} \), then \( M \) is an extrinsic sphere in \( \tilde{M} \). Conversely if \( M \) is an extrinsic sphere in \( \tilde{M} \), then every circle in \( M \) is a circle in \( \tilde{M} \).

In this paper, we study relations between isometric immersions and helices. We set \( \tilde{\sigma} = f \circ \sigma \) for a curve \( \sigma \) in \( M \). Let \( p \) be a point of \( M \) and \( d \geq 2 \). Let \( \lambda_1, \ldots, \lambda_{d-1} \) be positive constants. We consider the following conditions \( (C_1), (C_2) \) and \( (C_3) \):

\[
(C_1) \quad \begin{cases} 
\text{The first curvature } \tilde{\lambda}_1 \text{ of } \tilde{\sigma} \text{ is constant along } \tilde{\sigma} \text{ for every helix } \sigma \\
\text{of order } d \text{ through } p \text{ in } M \text{ with the } i\text{-th curvature } \lambda_i \ (1 \leq i \leq d - 1), \\
(C_1) \text{ holds and the second curvature } \tilde{\lambda}_2 \text{ of } \tilde{\sigma} \text{ is constant along } \tilde{\sigma} \\
\text{for every helix } \sigma \text{ of order } d \text{ though } p \text{ in } M \text{ with the } i\text{-th curvature } \lambda_i \ (1 \leq i \leq d - 1), 
\end{cases}
\]

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(C2) holds and the second curvature $\lambda_2$ of $\sigma$ is independent of the choice of helix $\sigma$ of order $d$ through $p$ in $M$ with the $i$-th curvature $\lambda_i$

$\{1 \leq i \leq d - 1\}$.

The result of Nomizu and Yano is given under the condition (C1) in the case where $d = 2$ and $\sigma$ is a circle for every circle $\sigma$. In Section 1, we give notations and equations which are used in this paper. In section 2, we obtain some results under the condition (C1). In Section 3, we treat the conditions (C2) and (C3). In Section 4, we study some curves under the condition (C2) where $\tilde{M}$ is of constant curvature.

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§ 1. Preliminaries

In this paper, the differentiability of all geometric objects will be $C^\infty$. Let $f: M \to \tilde{M}$ be an isometric immersion of an $n$-dimensional connected Riemannian manifold $M$ into an $m$-dimensional Riemannian manifold $\tilde{M}$. For all local formulas and computations, we may assume $f$ as an imbedding and thus we shall often identify $p \in M$ with $f(p) \in \tilde{M}$. The tangent space $T_pM$ is identified with a subspace $f_*(T_pM)$ of $T_p\tilde{M}$ where $f_*$ is the differential map of $f$. Letters $X$, $Y$ and $Z$ (resp. $\xi, \eta$ and $\zeta$) vector fields tangent (resp. normal) to $M$. We denote the tangent bundle of $M$ (resp. $\tilde{M}$) by $TM$ (resp. $T\tilde{M}$), unit tangent bundle of $M$ by $U_M$ and the normal bundle by $T^\perp_M$. Let $\nabla$ and $\tilde{\nabla}$ be the Levi-Civita connections of $\tilde{M}$ and $M$, respectively. Then the Gauss formula is given by

$$\tilde{\nabla}_XY = \nabla_XY + h(X,Y),$$

where $h$ denotes the second fundamental form. The Weingarten formula is given by

$$\tilde{\nabla}_X\xi = -A_\xi X + \nabla^\perp_X\xi,$$

where $A$ denotes the shape operator and $\nabla^\perp$ the normal connection. Clearly $A$ is related to $h$ as $\langle A_\xi X, Y \rangle = \langle h(X,Y), \xi \rangle$, where $\langle , \rangle$ denotes the Riemannian metrics of $M$ and $\tilde{M}$. We put $||\tilde{x}|| = \sqrt{\langle \tilde{x}, \tilde{x} \rangle}$ for $\tilde{x} \in T\tilde{M}$. For the second fundamental form and the shape operator, we define their covariant derivatives by

$$(Dh)(Z,X,Y) = \nabla^\perp_Z(h(X,Y)) - h(\nabla_ZX,Y) - h(X,\nabla_Y Z),$$

$$(DA)_\zeta(Y,X) = \nabla_Y(A_\zeta X) - A_{\nabla^\perp_Y \zeta}X - A_\zeta(\nabla_Y X).$$
Furthermore we define the $k$-th covariant derivative of $h$ as follows:

$$(D^k h)(X_1, X_2, \ldots, X_{k+2}) = \nabla_{X_1}((D^{k-1} h)(X_2, \ldots, X_{k+2}))$$

$$- \sum_{i=2}^{k+2} (D^{k-1} h)(X_2, \ldots, \nabla_{X_i} X_i, \ldots, X_{k+2})$$

where $k \geq 1$ and $D^0 h = h$. If, for the non-negative integers $i_1, i_2, \ldots, i_j$ ($j \geq 1$) satisfying that $i_1 + i_2 + \cdots + i_j = k + 2$ ($k \geq 0$), $X_1 = X_2 = \cdots = X_i = X$, $X_{i+1} = \cdots = X_{i+j} = Y$, $X_{i+j+1} = \cdots = X_{i+2j} = Z$, then a normal vector $(D^k h)(X_1, X_2, \ldots, X_{k+2})$ is written as $(D^k h)(X^i, Y^i, Z^i)$. Moreover a tangent vector $(DA)_\xi(X, X)$ will be written as $(DA)_\xi(X^2)$. The submanifold $M$ in $\mathcal{M}$ is said to be isotropic at $p \in M$ of a constant normal curvature $\mu$ if the normal vector $h(x^2)$ satisfies

$$\langle h(x^2), h(x^2) \rangle = \mu^2 \langle x, x \rangle^2$$

for every $x \in T_p M$. The above isotropic condition is equivalent with

$$(1.1) \quad \mathcal{S} \langle h(x, y), h(z, w) \rangle = \mathcal{S} \mu^2 \langle x, y \rangle \langle z, w \rangle$$

for $x, y, z, w \in T_p M$, where $\mathcal{S}$ denote the cyclic sum with respect to $x, y$ and $z$. (cf. B. O'Neill [5]). If there exists a non-negative function $\mu$ on $M$ such that $M$ is isotropic at $p$ of the constant normal curvature $\mu(p)$ for every point of $M$, then $M$ is said to be an isotropic submanifold. In particular, when $\mu$ is constant on $M$, $M$ is said to be constant isotropic. The mean curvature vector field $H$ of $M$ is defined by

$$H := \frac{1}{n} \sum_{i=1}^{n} h(e_i^2),$$

where $e_1, \ldots, e_n$ is an orthonormal frame at each point of $M$. If the second fundamental form $h$ satisfies $h(X, Y) = \langle X, Y \rangle H$, then $M$ is called a totally umbilical submanifold. The mean curvature vector field $H$ is said to be parallel if $\nabla H = 0$. A totally umbilical submanifold with the parallel mean curvature vector field is called an extrinsic sphere. If the second fundamental form $h$ vanishes identically, then we call $M$ a totally geodesic submanifold of $\mathcal{M}$.

Next we shall define a helix of order $d$ in a Riemannian manifold $N$. Let $\sigma : I \to N(s \mapsto \sigma(s))$ be a smooth curve in $N$, where $I$ is an open interval of the real line $\mathbb{R}$. We denote the tangent vector field $d\sigma/ds$ of $\sigma$ by $v_1$. We call $s$ a $d$-regular point of $\sigma$ if $\dim \operatorname{Span}\{\nabla_{v_1}^k v_1(s) | k = 0, \ldots, d-1\} = d$ where $\nabla_{v_1}^0 v_1 = v_1$ and $\nabla_{v_1}^k v_1 = \nabla_{v_1}(\nabla_{v_1}^{k-1} v_1)$ for $k \geq 1$. If every $s \in I$ is a $d$-regular point of $\sigma$, then $\sigma$
is said to be a \textit{d-regular curve}. Note that 1-regular curve is a usual regular curve. Hereafter, in this paper, we assume that all curves are regular and parametrized by arc length. If \( \sigma \) is a \( d \)-regular curve, then we put

\[
\begin{align*}
\{ & v_0 := 0, \quad w_0 := v_1, \quad \lambda_0 := 1, \\
& v_i := \frac{w_{i-1}}{\lambda_{i-1}}, \quad w_i := \nabla_{v_i}v_i + \lambda_{i-1}v_{i-1} \quad \text{and} \quad \lambda_i := ||w_i|| \quad \text{for} \quad 1 \leq i \leq d.
\end{align*}
\]

We call \( \lambda_i \) (\( 1 \leq i \leq d \)) (resp. \( w_i \)) the \( i \)-th curvature (resp. the \( i \)-th curvature vector field) and \( v_i \) \( (2 \leq i \leq d) \) the \((i-1)\)-th normal vector field. If \( \sigma \) is a \( d \)-regular curve and the \( d \)-th curvature \( \lambda_d \) of \( \sigma \) vanishes on \( I \), then we call such a curve a \textit{curve of order} \( d \) and \( v_1, \ldots, v_d \) the \textit{Frenet frame field}. Note that a curve of order one is a geodesic. In the case where \( \sigma \) is a curve of order \( d \), we put

\[
\begin{align*}
& v_i := 0, \quad w_i := 0 \quad \text{and} \quad \lambda_i := 0 \quad \text{for} \quad i > d.
\end{align*}
\]

From (1.2) and (1.3), we have the following Frenet formula of \( \sigma \)

\[
\nabla_{v_i}v_j + \lambda_{j-1}v_{j-1} = \lambda_j v_{j+1}
\]

for \( j \geq 1 \). If \( \sigma \) is a curve of order \( d \) and \( \lambda_i \) are constant along \( \sigma \), then we call this a \textit{helix of order} \( d \). Note that a helix of order two is a circle.

\section{Helices in a Riemannian submanifold}

Let \( f : M \rightarrow \tilde{M} \) be an isometric immersion of an \( n \)-dimensional connected Riemannian manifold into an \( m \)-dimensional Riemannian manifold \( \tilde{M} \). Let \( \sigma \) be a helix of order \( d \) in \( \tilde{M} \) with the \( i \)-th curvature \( \lambda_i (1 \leq i \leq d-1) \) and the Frenet frame field \( v_1, \ldots, v_d \). We set \( \tilde{\sigma} := f \circ \sigma \). We have \( \tilde{v}_1 = d\tilde{\sigma}/ds = v_1 \). From the Gauss formula and the Frenet formula of \( \sigma \), we get \( \tilde{\nabla}_{v_1}v_1 = \lambda_1 v_2 + h(v_1^2) \). Since \( \tilde{\sigma} \) is a regular curve, we have

\[
\tilde{w}_1 = \lambda_1 v_2 + h(v_1, v_1), \quad \tilde{\lambda}_1^2 = \lambda_1^2 + \langle h(v_1^2), h(v_1^2) \rangle,
\]

where \( \tilde{w}_1 \) is the first curvature vector field of \( \tilde{\sigma} \). First we prove the following lemma.

\begin{lemma}
Let \( d \geq 1 \) and \( \lambda_1, \ldots, \lambda_{d-1} \) be positive constants. Let \( \mu \) be non-negative constant and \( p \in \tilde{M} \). Then the following conditions are equivalent:

(a) The first curvature \( \tilde{\lambda}_1 \) of \( \tilde{\sigma} \) at \( p \) is equal to \( \mu \) for every helix \( \sigma \) of order \( d \) through \( p \) in \( \tilde{M} \) with the \( i \)-th curvature \( \lambda_i (1 \leq i \leq d-1) \),

(b) \( \tilde{M} \) is isotropic at \( p \) in \( \tilde{M} \) of the normal curvature \( \sqrt{\mu^2 - \lambda_1^2} \).
\end{lemma}
Proof. Suppose that (a) holds. Let \( x_0 \) be any unit tangent vector at \( p \) in \( M \). We take a helix \( \sigma \) of order \( d \) in \( M \) with the \( i \)-th curvature \( \lambda_i (1 \leq i \leq d - 1) \) satisfying that \( \sigma(0) = p \) and \( v_1(0) = x_0 \) where \( v_1 \) is the tangent vector field of \( \sigma \). From (2.1), we have \( \mu^2 = \lambda_1^2 + \langle h(x_0^2), h(x_0^2) \rangle \). Hence we get \( \langle h(x_0^2), h(x_0^2) \rangle = \mu^2 - \lambda_1^2 \) for every \( x \in U_p M \). Therefore we see that \( M \) is isotropic at \( p \). Hence we get (b).

Suppose that (b) holds. Let \( x_0 \) be any unit tangent vector at \( p \) in \( M \). We take a helix \( \sigma \) of order \( d \) in \( M \) with the \( i \)-th curvature \( \lambda_i \) satisfying that \( \sigma(0) = p \) and \( v_1(0) = x_0 \) where \( v_1 \) is the tangent vector field of \( \sigma \). Set \( \tilde{\lambda}_1 \) the first curvature of \( \sigma \).

From (2.1), we have
\[
\tilde{\lambda}_1^2(0) = \lambda_1^2 + \langle h(x_0^2), h(x_0^2) \rangle = \lambda_1^2 + (\mu^2 - \lambda_1^2) = \mu^2.
\]
Hence we get (a).

Remark. If \( M \) is isotropic at \( p \) of a normal curvature \( \mu \), then it is clear from (1.1) that
\[
A_{h(x^2)}x = \mu^2 x \quad \text{for} \quad x \in U_p M.
\]

Let \( p \) be a point of \( M \), \( d \geq 2 \) and \( \lambda_1, \ldots, \lambda_{d-1} \) positive constants. We consider the following the condition \((C_1)\):

\[
(C_1) \begin{cases} 
\text{The first curvature } \tilde{\lambda}_1 \text{ of } \sigma \text{ is constant along } \sigma \text{ for every helix } \sigma \\
\text{of order } d \text{ through } p \text{ in } M \text{ with the } i \text{-th curvature } \lambda_i (1 \leq i \leq d - 1). 
\end{cases}
\]

From Lemma 2.1, we obtain the following Lemma.

Lemma 2.2. Let \( d \geq 2 \) and \( \lambda_1, \ldots, \lambda_{d-1} \) be positive constants. Let \( p \) be a point of \( M \) satisfying \((C_1)\). Then \( M \) is isotropic at \( p \) of the normal curvature \( \sqrt{\lambda_1^2 - \lambda_{d-1}^2} \) (i.e., \( \tilde{\lambda}_1 \) is independent of the choice of \( \sigma \)). Moreover we get

\[
(2.2) \quad \langle h(v, z), (Dh)(y, x, w) \rangle + \langle h(w, z), (Dh)(y, x, v) \rangle \\
+ \langle h(x, z), (Dh)(y, w, v) \rangle + \langle h(w, v), (Dh)(y, x, z) \rangle \\
+ \langle h(x, v), (Dh)(y, w, z) \rangle + \langle h(x, w), (Dh)(y, v, z) \rangle = 0
\]

for every \( x, y, z, v, w \in T_p M \).

Proof. Let \( x \) and \( y \) be any orthonormal tangent vectors at \( p \) in \( M \). We take a helix \( \sigma \) of order \( d \) in \( M \) with the \( i \)-th curvature \( \lambda_i \) satisfying that \( \sigma(0) = p \),
$v_1(0) = x$ and $v_2(0) = y$ where $v_1$ (resp. $v_2$) is the tangent vector field of $\sigma$ (resp. the first normal vector field of $\sigma$). From (2.1), we get $\tilde{\lambda}_1^2 = \lambda_1^2 + \langle h(v_1^2), h(v_2^2) \rangle$.

Applying $\tilde{\nabla}_{v_1}$ to this equation and using the Frenet formula of $\sigma$, we get

\[(2.3) \quad \langle (Dh)(v_1^2), h(v_2^2) \rangle + 2\lambda_1 \langle h(v_1), h(v_2) \rangle = 0.\]

Moreover, applying $\tilde{\nabla}_{v_1}$ to (2.3) and using the Frenet formula of $\sigma$, we get

\[(2.4) \quad \langle (D^2h)(v_1^4), h(v_2^2) \rangle + \langle (Dh)(v_1^3), (Dh)(v_1^3) \rangle + \lambda_1 \langle (Dh)(v_2, v_1^2), h(v_1^2) \rangle + 4\lambda_1 \langle (Dh)(v_1^2, v_2), h(v_1^2) \rangle + 4\lambda_1 \langle (Dh)(v_1^2), h(v_2) \rangle + 2\lambda_1 \langle h(v_1), h(v_2) \rangle + 2\lambda_2 \langle h(v_1^2), h(v_1^2) \rangle + 2\lambda_1 \lambda_2 \langle h(v_1^2), h(v_1^3) \rangle = 0.\]

From (2.3), we get

\[\langle (Dh)(x^3), h(x^2) \rangle + 2\lambda_1 \langle h(x, y), h(x^2) \rangle = 0.\]

Since $x$ and $-y$ are orthonormal tangent vectors and $\lambda_1 > 0$, we obtain that

\[\langle (Dh)(x^3), h(x^2) \rangle = \langle h(x, y), h(x^2) \rangle = 0.\]

Hence we have $\langle h(x^2), h(x, y) \rangle = 0$ for every $x, y \in U_pM$ such that $\langle x, y \rangle = 0$ and

\[(2.5) \quad \langle (Dh)(x^3), h(x^2) \rangle = 0\]

for every $x \in T_pM$. Therefore we get $M$ is isotropic at $p$ of the normal curvature $\sqrt{\lambda_1^2 - \lambda_2^2}$. From Lemma 2.1, we see that $\tilde{\lambda}_1$ is independent of the choice of $\sigma$.

Also, from (1.1) and (2.4), we get

\[\langle (D^2h)(x^4), h(x^2) \rangle + \langle (Dh)(x^3), (Dh)(x^3) \rangle + \lambda_1 \langle (Dh)(y, x^2), h(x^2) \rangle + 4\lambda_1 \langle (Dh)(x^2, y), h(x^2) \rangle + 4\lambda_1 \langle (Dh)(x^3), h(x, y) \rangle = 0\]

for every $x, y \in U_pM$ such that $\langle x, y \rangle = 0$. Since $x$ and $-y$ are orthonormal and $\lambda_1 > 0$, we get

\[(2.6) \quad \langle (Dh)(y, x^2), h(x^2) \rangle + 4\langle (Dh)(x^2, y), h(x^2) \rangle + 4\langle (Dh)(x^3), h(x, y) \rangle = 0\]

for every $x, y \in U_pM$ such that $\langle x, y \rangle = 0$. From (2.5), we have

\[(2.7) \quad \langle (Dh)(y, x^2), h(x^2) \rangle + 2\langle (Dh)(x^2, y), h(x^2) \rangle + 2\langle (Dh)(x^3), h(x, y) \rangle = 0\]

for every $x, y \in T_pM$. From (2.6) and (2.7), it follows that
for every \( x, y \in U_p M \) such that \( \langle x, y \rangle = 0 \). Hence, from (2.5), we see that
\[
\langle h(x^2), (Dh)(y, x^2) \rangle = 0
\]
for every \( x, y \in T_p M \).

Since \( h \) is symmetric, we have (2.2) for any tangent vectors \( x, y, z, v \) and \( w \) at \( p \).

From Lemma 2.2, we get

**Proposition 2.3.** Let \( M \) be an \( n \)-dimensional connected Riemannian submanifold in an \( m \)-dimensional Riemannian manifold \( \tilde{M} \) isometrically immersed by \( f \) and \( n \geq 2 \). Let \( d \geq 2 \) and \( \lambda_1, \ldots, \lambda_{d-1} \) be positive constants. If the condition \((C_1)\) holds at every point of \( M \), then \( M \) is a constant isotropic submanifold of \( \tilde{M} \).

**Proof.** By Lemma 2.2, we see that \( M \) is an isotropic submanifold. Then there exists a non-negative function \( \mu \) on \( M \) such that \( M \) is isotropic at \( p \) of the constant normal curvature \( \mu(p) \) for every point \( p \) of \( M \). We shall show that the derivative of \( \mu^2 \) vanishes on \( M \). Let \( p \in M \) and \( x \in U_p M \) be arbitrarily fixed. For a unit vector field \( Y \) on a neighborhood of \( p \), we have
\[
x \mu^2 = x \langle h(Y^2), h(Y^2) \rangle = 2 \langle (Dh)(x, Y^2), h(Y^2) \rangle_{\text{at } p} + 4 \langle h(\nabla X Y, Y), h(Y^2) \rangle_{\text{at } p}.
\]
Since the equation (2.2) holds and \( \langle \nabla X Y, Y \rangle = 0 \), we get \( x \mu^2 = 0 \). Hence we see that \( M \) is constant isotropic.

**§3. The discriminant of the second fundamental form**

Let \( M, \tilde{M} \) and \( f \) be as in §2. Let \( \sigma \) be a helix of order \( d \) in \( M \) with the \( i \)-th curvature \( \lambda_i (1 \leq i \leq d - 1) \) and the Frenet frame field \( v_1, \ldots, v_d \). Let \( \tilde{\lambda}_i (1 \leq i) \) be the \( i \)-th curvature of \( \tilde{\sigma} \). By a routine calculation, we have the following lemma.

**Lemma 3.1.** The tangent vector field \( \tilde{v}_1 \) and the first curvature vector field \( \tilde{w}_1 \) of \( \tilde{\sigma} \) are given by
\[
\tilde{v}_1 = v_1, \quad \tilde{w}_1 = \lambda_1 v_2 + h(v_1^2).
\]
If \( \tilde{\lambda}_1 \) is constant along \( \tilde{\sigma} \) then the second curvature vector field \( \tilde{w}_2 \) of \( \tilde{\sigma} \) is given by
\[
\tilde{\lambda}_1 \tilde{w}_2 = (\tilde{\lambda}_1^2 - \lambda_1^2) v_1 + \lambda_1 \lambda_2 v_3 - A_{h(v_1)} v_1 + 3 \lambda_1 h(v_1, v_2) + (Dh)(v_1^3).
\]
Moreover, If $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are constant along $\tilde{\sigma}$, then the third curvature vector field $\tilde{w}_3$ of $\tilde{\sigma}$ is given by

\begin{equation}
\tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{w}_3 = \lambda_1 (\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 - \lambda_1^2 - \lambda_2^2) v_2 + \lambda_1 \lambda_2 \lambda_3 v_4 - (DA)_{h(v_i)} (v_i^2) \\
- 5\lambda_1 A_{h(v_i)} v_1 - \lambda_1 A_{h(v_i)} v_2 - 2A_{(Dh)(v_i^2)} v_1 - h(v_1, A_{h(v_i)} v_1) \\
+ (\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 - 4\lambda_1^2) h(v_i^2) + 3\lambda_1^2 h(v_i^2) + 4\lambda_1 \lambda_2 h(v_1, v_2) \\
+ 5\lambda_1 (Dh)(v_i^2, v_2) + \lambda_1 (Dh)(v_2, v_i^2) + (D^2 h)(v_i^4).
\end{equation}

We prove the following lemma.

**Lemma 3.2.** Let $p$ be a point of $M$, $d \geq 2$ and $\lambda_1, \ldots, \lambda_{d-1}$ positive constants. If, for every helix $\sigma$ of order $d$ through $p$ in $M$ with the $i$-th curvature $\lambda_i$ ($1 \leq i \leq d - 1$),

\begin{equation}
v_1 \langle h(v_1, v_2), (Dh)(v_i^2) \rangle = 0 \text{ at } p
\end{equation}

where $v_1$ (resp. $v_2$) is the tangent vector field of $\sigma$ (resp. the first normal vector field of $\sigma$), then we have

\begin{equation}
\langle (Dh)(x^2, y), (Dh)(x^3) \rangle + \langle (D^2 h)(x^4), h(x, y) \rangle = 0
\end{equation}

for every $x, y \in U_p M$ such that $\langle x, y \rangle = 0$.

**Proof.** Let $x$ and $y$ be any orthonormal tangent vectors at $p$ in $M$. We take a helix $\sigma$ of order $d$ in $M$ with the $i$-th curvature $\lambda_i$ satisfying that $\sigma(0) = p$, $v_1(0) = x$ and $v_2(0) = y$. By assumption, we have

$$0 = v_1 \langle h(v_1, v_2), (Dh)(v_i^2) \rangle|_{s=0}$$

$$= \langle (Dh)(v_1^2, v_2), (Dh)(v_i^2) \rangle|_{s=0} + \langle h(\nabla_{v_1} v_1, v_2), (Dh)(v_i^2) \rangle|_{s=0}$$

$$+ \langle h(v_1, \nabla_{v_1} v_2), (Dh)(v_i^2) \rangle|_{s=0} + \langle h(v_1, v_2), (D^2 h)(v_i^4) \rangle|_{s=0}$$

$$+ \langle h(v_1, v_2), (Dh)(\nabla_{v_1} v_1, v_i^2) \rangle|_{s=0} + 2\langle h(v_1, v_2), (Dh)(v_i^2, \nabla_{v_1} v_1) \rangle|_{s=0}$$

$$= \langle (Dh)(x^2, y), (Dh)(x^3) \rangle + \lambda_1 \langle h(y^2), (Dh)(x^3) \rangle - \lambda_1 \langle h(x^2), (Dh)(x^3) \rangle$$

$$+ \lambda_2 \langle h(x, v_3(0)), (Dh)(x^3) \rangle + \langle h(x, y), (D^2 h)(x^4) \rangle$$

$$+ \lambda_1 \langle h(x, y), (Dh)(y, x^2) \rangle + 2\lambda_1 \langle h(x, y), (Dh)(x^2, y) \rangle$$
where $v_3$ is the second normal vector field of $\sigma$. If $d = 2$, then $v_3 = 0$. Since $x$ and $-y$ are orthonormal, we have (3.4). If $d \geq 3$, then we can take a unit vector $z(\in T_pM)$ satisfying that $v_3(0) = z$. Also since $x, -y$ and $z$ are orthonormal, we have (3.4).

Let $\sigma$ be a helix of order $d$ in $M$ and $d \geq 2$. From (2.1), we have $\tilde{\lambda}_1 \geq \lambda_1 > 0$ where $\tilde{\lambda}_1$ (resp. $\lambda_1$) is the first curvature of $\tilde{\sigma}$ (resp. the first curvature of $\sigma$). Thus $\tilde{\sigma}$ is a 2-regular curve. Let $p$ be a point of $M$, $d \geq 2$ and $\lambda_1 \cdots \lambda_{d-1}$ positive-constants. We consider the following conditions (C2) and (C3):

\begin{align*}
(C_1) & \text{ holds and the second curvature } \tilde{\lambda}_2 \text{ of } \tilde{\sigma} \text{ is constant along } \tilde{\sigma} \\
(C_2) & \text{ for every helix } \sigma \text{ of order } d \text{ through } p \text{ in } M \text{ with the } i\text{-th curvature } \lambda_i \\text{ (}1 \leq i \leq d-1\text{),} \\
(C_3) & \text{ holds and the second curvature } \tilde{\lambda}_2 \text{ of } \tilde{\sigma} \text{ is independent of the choice of helix } \sigma \text{ of order } d \text{ through } p \text{ in } M \text{ with the } i\text{-th curvature } \lambda_i \\text{ (}1 \leq i \leq d-1\text{).}
\end{align*}

For $x \in UM$, we set

$$v(x) := \langle (Dh)(x^3), (Dh)(x^3) \rangle.$$

We prove the following lemma.

**Lemma 3.3.** Let $d \geq 2$ and $\lambda_1, \ldots, \lambda_{d-1}$ be positive constants. Let $p$ be a point of $M$ satisfying (C2). Then $v$ is constant on $U_pM$ if and only if (3.4) holds for every $x, y \in U_pM$ such that $\langle x, y \rangle = 0$. Moreover, we get

\begin{equation}
15\lambda_1^2 \langle (Dh)(x^2, y), h(x, y) \rangle + 3\lambda_1 \langle (Dh)(y, x^2), h(x, y) \rangle + 3\lambda_2^2 \langle (Dh)(x^3), h(y^2) \rangle + \langle (D^2h)(x^4), (Dh)(x^3) \rangle = 0
\end{equation}

for every $x, y \in U_pM$ such that $\langle x, y \rangle = 0$. Moreover, if $d \geq 3$, then we have

\begin{equation}
\langle (Dh)(x^3), h(x, y) \rangle = \langle (Dh)(y, x^2), h(x^2) \rangle = \langle (Dh)(x^2, y), h(x^2) \rangle = 0
\end{equation}

for every $x, y \in T_pM$.

**Proof.** Let $x$ and $y$ be any orthonormal tangent vectors at $p$ in $M$. We take a helix $\sigma$ of order $d$ in $M$ with the $i$-th curvature $\lambda_i$ satisfying that $\sigma(0) = p$,
\( v_1(0) = x \) and \( v_2(0) = y \) where \( v_1 \) (resp. \( v_2 \)) is the tangent vector field of \( \sigma \) (resp. the first normal vector field of \( \sigma \)). Since (2.2), (3.1) and (3.2) hold and \( M \) is isotropic at \( p \) by Lemma 2.2, we obtain

\[
0 = \langle \hat{\lambda}_1 \hat{w}_2, \hat{\lambda}_1 \hat{\lambda}_2 \hat{w}_3 \rangle|_{\gamma=0} \\
= 9\lambda_1^2 \lambda_2 \langle h(x, y), h(x, v_3(0)) \rangle + 3\lambda_1 \lambda_2 \langle (D^2h)(x^3), h(x, v_3(0)) \rangle \\
+ 15\lambda_1^2 \langle (Dh)(x^2, y), h(x, y) \rangle + 3\lambda_1^2 \langle (Dh)(y, x^2), h(x, y) \rangle \\
+ 3\lambda_1 \langle (D^2h)(x^4), h(x, y) \rangle + 3\lambda_1^2 \langle (Dh)(x^3), h(y^2) \rangle \\
+ 5\lambda_1 \langle (Dh)(x^2, y), (Dh)(x^3) \rangle + \lambda_1 \langle (Dh)(y, x^2), (Dh)(x^3) \rangle \\
+ \langle (D^2h)(x^4), (Dh)(x^3) \rangle
\]

where \( v_3 \) is the second normal vector field of \( \sigma \).

If \( d = 2 \), then \( v_3 = 0 \). We have

\[
(3.7) \quad 15\lambda_1^2 \langle (Dh)(x^2, y), h(x, y) \rangle + 3\lambda_1^2 \langle (Dh)(y, x^2), h(x, y) \rangle \\
+ 3\lambda_1 \langle (D^2h)(x^3), h(y^2) \rangle + 3\lambda_1 \langle (D^2h)(x^4), h(x, y) \rangle \\
+ 5\lambda_1 \langle (Dh)(x^2, y), (Dh)(x^3) \rangle + \lambda_1 \langle (Dh)(y, x^2), (Dh)(x^3) \rangle \\
+ \langle (D^2h)(x^4), (Dh)(x^3) \rangle = 0.
\]

If \( d \geq 3 \), we can take a unit vector \( z(\in T_p M) \) satisfying \( v_3(0) = z \). Since \( x, y \) and \( -z \) are orthonormal, we get (3.7) and

\[
(3.8) \quad 9\lambda_1^2 \lambda_2 \langle h(x, y), h(x, z) \rangle + 3\lambda_1 \lambda_2 \langle (Dh)(x^3), h(x, z) \rangle = 0.
\]

In any case, we see that (3.7) holds for every \( x, y \in U_p M \) such that \( \langle x, y \rangle = 0 \). Since \( x \) and \( -y \) are orthonormal and \( \lambda_1 > 0 \), we obtain that (3.5) and

\[
3\langle h(x, y), (D^2h)(x^4) \rangle + \langle (Dh)(x^3), (Dh)(x^2, y) \rangle \\
= 2\langle (Dh)(x^3), (Dh)(x^2, y) \rangle + \langle (Dh)(x^3), (Dh)(y, x^2) \rangle
\]

for every \( x, y \in U_p M \) such that \( \langle x, y \rangle = 0 \). If (3.4) holds, then we have

\[
2\langle (Dh)(x^2, y), (Dh)(x^3) \rangle + \langle (Dh)(y, x^2), (Dh)(x^3) \rangle = 0
\]

for every \( x, y \in U_p M \) such that \( \langle x, y \rangle = 0 \). Hence we get \( v \) is constant on \( U_p M \). The converse is rather clear.
Helices and isometric immersions

Here, we assume that \(d \geq 3\). Since (3.8) holds for \(x, y, z\) and \(\lambda_1\lambda_2 > 0\), we have \(\langle (Dh)(x^3), h(x, z) \rangle = 0\) for every \(x, y \in U_pM\) such that \(\langle x, z \rangle = 0\). From this equation and (2.2), we have (3.6).

Let \(p\) be a point of \(M\). The discriminant \(\Delta\) at \(p\) of the second fundamental form \(h\) is given by

\[
\Delta_{xy} = \frac{\langle h(x^2), h(y^2) \rangle - \|h(x,y)\|^2}{\|x\|^2 \|y\|^2 - \langle x, y \rangle^2}
\]

for linearly independent tangent vectors \(x, y \in T_pM\).

We assume that \(p\) is a point of \(M\) satisfying (C2). We take a helix \(\sigma\) of order \(d\) through \(p\) and put \(v_1(0) = x\) and \(v_2(0) = y\) where \(d \geq 2\). From (2.3) and the fact that \(M\) is isotropic at \(p\), we get

\[
9\lambda_1^2 \langle h(x, y), h(x, y) \rangle + 6\lambda_1 \langle (Dh)(x^3), h(x, y) \rangle + \nu(x) + \lambda_1^2 \lambda_2^2 - \frac{\lambda_1^2 \lambda_2^2}{\lambda_1 + \lambda_2} = 0.
\]

for \(\tilde{\alpha}\). In particular, if (3.6) holds, then we get

\[
9\lambda_1^2 \langle h(x, y), h(x, y) \rangle + \nu(x) + \lambda_1^2 \lambda_2^2 - \frac{\lambda_1^2 \lambda_2^2}{\lambda_1 + \lambda_2} = 0.
\]

Moreover, from (1.1), we get

\[
\Delta_{xy} = (\lambda_1^2 - \lambda_2^2) - \frac{1}{3\lambda_1^2} (\lambda_1^2 \lambda_2^2 - \lambda_1^2 \lambda_2^2 - \nu(x)).
\]

From Lemma 3.2 and Lemma 3.3, we have the following theorem:

**Theorem 3.4.** Let \(M\) be an \(n\)-dimensional connected Riemannian submanifold in an \(m\)-dimensional Riemannian manifold \(\tilde{M}\) isometrically immersed by \(f\) and \(n \geq 3\). Let \(d \geq 3\) and \(\lambda_1, \ldots, \lambda_{d-1}\) be positive constants. Suppose that the condition (C1) holds at every point of \(M\). Let \(p\) be a point of \(M\). If the condition (C2) holds at \(p\), then \(\nu\) is constant on \(U_pM\). Moreover the discriminant \(\Delta\) at \(p\) is constant if and only if the condition (C3) holds at \(p\).

In case of \(d = 2\), we shall prove that (3.6) holds at \(p\) under the condition (C3). We have the following lemma.

**Lemma 3.5.** Let \(d = 2\) and \(\lambda_2\) be a positive constant. Let \(p\) be a point of \(M\) satisfying (C3). Then we have (3.6) for every \(x, y \in T_pM\). Moreover we get (3.10) and (3.11) for every \(x, y \in U_pM\) such that \(\langle x, y \rangle = 0\).
Proof. We have (3.9) for any \( x, y \in U_pM \) such that \( \langle x, y \rangle = 0 \). Since \( x \) and \(-y\) are orthonormal and \( p \) is a point satisfying (C3), we obtain \( \lambda_1 \langle (Dh)(x^3), h(x, y) \rangle = 0 \) and (3.10). From (1.1), we obtain (3.11). Since \( \lambda_1 > 0 \) and (2.2) holds, we get (3.6).

From the definition of discriminant, we have the following theorem.

Theorem 3.6. Let \( M \) be an \( n \)-dimensional connected Riemannian submanifold in an \( m \)-dimensional Riemannian manifold \( \tilde{M} \) isometrically immersed by \( f \) and \( n \geq 3 \). Let \( d \geq 2 \) and \( \lambda_1, \ldots, \lambda_{d-1} \) be positive constants. Let \( p \) be a point of \( M \) satisfying the condition (C3). Then \( v \) is constant on \( U_pM \) and the discriminant \( \Delta \) at \( p \) is constant.

Proof. Let \( x, y, z \) be orthonormal in \( T_pM \). Set \( x(\theta) = \cos \theta x + \sin \theta y \). From (3.11), we get

\[
(3.12) \quad v(x(\theta)) = \langle (Dh)(x(\theta)^3), (Dh)(x(\theta)^3) \rangle = \langle (Dh)(x^3), (Dh)(x^3) \rangle = v(z)
\]

Differentiating (3.12) at \( \theta = 0 \), we see that

\[
\langle (Dh)(y, x^2), (Dh)(x^3) \rangle + 2\langle (Dh)(x^2, y), (Dh)(x^3) \rangle = 0.
\]

Therefore we have \( v \) is constant on \( U_pM \). It is clear that the discriminant \( \Delta \) at \( p \) is constant.

In case of \( n = 2 \), from Lemma 2.2, we get the following lemma.

Lemma 3.7. Let \( n = 2 \) and \( d = 2 \). Let \( \lambda_1 \) be a positive constant and \( p \) a point of \( M \) satisfying (C1). Then the discriminant \( \Delta \) is constant at \( p \) and

\[
(3.13) \quad \|h(x, y)\|^2 = \frac{\lambda_1^2 - \lambda_2^2 - \Delta}{3} \quad \text{and} \quad \langle h(x^2), h(y^2) \rangle = \frac{\lambda_1^2 - \lambda_2^2 + 2\Delta}{3}
\]

for every \( x, y \in U_pM \) such that \( \langle x, y \rangle = 0 \). Thus \( \|h(x, y)\| \) and \( \langle h(x^2), h(y^2) \rangle \) are constant for every \( x, y \in U_pM \) such that \( \langle x, y \rangle = 0 \).

Proof. Let \( x, y \) be orthonormal in \( T_pM \). Set \( x(\theta) = \cos \theta x + \sin \theta y \) and \( y(\theta) = -\sin \theta x + \cos \theta y \). Since \( M \) is isotropic at \( p \), we get

\[
\frac{d}{d\theta} \Delta_{x(\theta)y(\theta)} = 4\langle h(y(\theta)^2), h(x(\theta), y(\theta)) \rangle - 4\langle h(x(\theta)^2), h(x(\theta), y(\theta)) \rangle = 0.
\]
Hence we get \( \Delta x(\theta)y(\theta) = \Delta x y \). From the definition of \( \Delta \), we get (3.13) for every \( x, y \in U_p M \) such that \( \langle x, y \rangle = 0 \).

From Theorem 3.6, Lemma 3.7 and Theorem 1 in [5], we get

**Corollary 3.8.** Let \( d \geq 2 \) and \( \lambda_1, \ldots, \lambda_{d-1} \) be positive constants. If (C3) holds for every point of \( M \) and \( m - n < (n + 2)(n - 1)/2 \), then \( M \) is a totally umbilic submanifold of \( \tilde{M} \). Moreover, at every point \( p \in M \), we get

\[
\langle H, H \rangle = \tilde{\lambda}_1^2 - \lambda_1^2, \\
\tilde{\lambda}_1^2 \tilde{\lambda}_2^2 - \lambda_1^2 \lambda_2^2 = \langle \nabla x H, \nabla x H \rangle
\]

for every \( x \in U_p M \) where \( H \) is the mean curvature vector field of \( M \).

**Remark.** In Corollary 3.8, we see that \( M \) is an extrinsic sphere if and only if \( \tilde{\lambda}_1^2 \tilde{\lambda}_2^2 = \lambda_1^2 \lambda_2^2 \). Then \( \tilde{\lambda}_2 \leq \lambda_2 \).

§ 4. Curves in a Riemannian manifold of constant curvature

Let \( M \) be an \( n \)-dimensional connected Riemannian submanifold in an \( m \)-dimensional Riemannian manifold \( \tilde{M} \) of constant curvature \( c \) isometrically immersed by \( f \). From the Codazzi equation, it is known that

\[
R(x, y)z = c\{\langle y, z \rangle x - \langle x, z \rangle y \} + A_{h(y, z)}x - A_{h(x, z)}y,
\]

(4.1)

\[
(Dh)(x, y, z) = (Dh)(y, x, z),
\]

(4.2)

\[
R^\perp(x, y, z) = h(x, A_x y) - h(A_x x, y)
\]

(4.3)

for \( x, y, z \in TM \) and \( \xi \in T^\perp M \) where \( R \) and \( R^\perp \) are the curvature tensor of \( V \) and \( V^\perp \). From (4.2) and Lemma 2.2, we get

**Lemma 4.1.** Let \( p \) be a point of \( M \), \( d = 2 \) and \( \lambda_1 \) a positive constant. If (C1) holds at \( p \), then we obtain (3.6) for every \( x, y \in T_p M \).

From Lemma 3.2, Lemma 3.3 and Lemma 4.1, we get the following theorem.

**Theorem 4.2.** Let \( M \) be an \( n \)-dimensional connected Riemannian submanifold in an \( m \)-dimensional Riemannian manifold \( \tilde{M} \) of constant curvature \( c \) isometrically
immersed by $f$ and $n \geq 2$. Let $d = 2$ and $\lambda_1$ be a positive constant. Suppose that the condition $(C_1)$ holds at every point of $M$. Let $p$ a point of $M$. If the condition $(C_2)$ holds at $p$, then $v$ is constant on $U_pM$ and the condition $(C_3)$ holds at $p$.

Let $p$ be a point of $M$ and $\alpha$ a constant. We define a $(0,6)$-tensor $F$ by

$$F(x, y, z, u, v, w) := \langle (Dh)(x, y, z), (Dh)(u, v, w) \rangle$$

$$- \alpha \frac{1}{9} \left\{ \langle y, z \rangle \langle x, u \rangle \langle v, w \rangle + \langle y, z \rangle \langle x, v \rangle \langle u, w \rangle + \langle x, z \rangle \langle y, w \rangle \langle u, v \rangle + \langle x, z \rangle \langle y, v \rangle \langle u, w \rangle + \langle x, y \rangle \langle z, v \rangle \langle u, w \rangle + \langle x, y \rangle \langle z, w \rangle \langle u, v \rangle \right\}$$

for $x, y, z, u, v, w \in T_pM$. We have the following Lemma 4.3. The proof of Lemma 4.3 is analogous to that of Lemma 2 in [5].

**Lemma 4.3.** Let $\tilde{M}$ be of constant curvature, $p$ a point of $M$ and $\alpha$ a constant. Then the following conditions are equivalent:

(a) $\langle (Dh)(x, x, x), (Dh)(x, x, x) \rangle = \alpha \langle x, x \rangle^3$ for every $x \in T_pM$,

(b) $F(x, y, z, u, v, w) + F(x, y, u, v, w, z) + F(x, y, v, w, z, u) + F(x, y, w, z, u, v)$

$$+ F(x, u, w, y, z, v) + F(x, z, v, y, u, w) + F(x, z, u, y, v, w)$$

$$+ F(x, v, w, y, z, u) + F(x, z, w, y, v, u) + F(x, v, u, y, z, w) = 0$$

for $x, y, z, u, v, w \in T_pM$.

Let $n = 2$. We assume that $p \in M$ is a point satisfying all conditions of Theorem 4.2. Let $N_1(p)$ be the first normal space at $p$ given by $\text{Span}\{h(x, y) | x, y \in T_pM\}$. Let $e_1, e_2$ be an orthonormal base of $T_pM$. Put

$$h_{ij} := h(e_i, e_j) \quad \text{for } 1 \leq i, j \leq 2,$$

$$Dh_{ijk} := (Dh)(e_i, e_j, e_k) \quad \text{for } 1 \leq i, j, k \leq 2.$$
From Lemma 4.3 and (3.6), we get
\[
\begin{cases}
\langle Dh_{111}, Dh_{111} \rangle = \langle Dh_{222}, Dh_{222} \rangle = v(p), \\
\langle Dh_{111}, Dh_{112} \rangle = 0, \\
\langle Dh_{111}, Dh_{222} \rangle + 9\langle Dh_{112}, Dh_{122} \rangle = 0,
\end{cases}
\]
(4.4)

\[
\begin{cases}
\langle Dh_{111}, h_{11} \rangle = \langle Dh_{222}, h_{22} \rangle = 0, \\
\langle Dh_{111}, h_{12} \rangle = \langle Dh_{112}, h_{11} \rangle = \langle Dh_{222}, h_{12} \rangle = \langle Dh_{122}, h_{22} \rangle = 0, \\
\langle Dh_{111}, h_{22} \rangle + 3\langle Dh_{112}, h_{12} \rangle = 0, \\
\langle Dh_{122}, h_{11} \rangle + \langle Dh_{112}, h_{12} \rangle = \langle Dh_{112}, h_{22} \rangle + \langle Dh_{122}, h_{12} \rangle = 0.
\end{cases}
\]
(4.5)

Let $K$ be the Gauss curvature of $M$. Then $K = c + \Delta$. From Lemma 2.2 and Theorem 1 in [5], we get
\[-2(\tilde{\lambda}_1^2 - \lambda_1^2) \leq \Delta(p) \leq \tilde{\lambda}_1^2 - \lambda_1^2,\]
\[\dim N_1(p) = 0 \Leftrightarrow \Delta(p) = \tilde{\lambda}_1^2 - \lambda_1^2 = 0 \text{ (i.e., } \tilde{\lambda}_1 = \lambda_1) \Leftrightarrow p \text{ is a geodesic point,}\]
\[\dim N_1(p) = 1 \Leftrightarrow \Delta(p) = \tilde{\lambda}_1^2 - \lambda_1^2 > 0 \Leftrightarrow p \text{ is a non-geodesic umbilic point,}\]
\[\dim N_1(p) = 2 \Leftrightarrow \Delta(p) = -2(\tilde{\lambda}_1^2 - \lambda_1^2) < 0 \Leftrightarrow p \text{ is a non-geodesic minimal point,}\]
\[\dim N_1(p) = 3 \Leftrightarrow -2(\tilde{\lambda}_1^2 - \lambda_1^2) < \Delta(p) < \tilde{\lambda}_1^2 - \lambda_1^2.\]

We shall prove the following Lemma.

**Lemma 4.4.** Let $n = 2$ and $m \leq 5$. Let $d = 2$ and $\lambda_1$ be a positive constant. We assume that (C_1) holds at every point of $M$. Let $p$ be a point of $M$. If (C_2) holds at $p$ and $2 \leq \dim N_1(p) \leq 3$, then $v(p) = 0$ (i.e., the second fundamental form $h$ is parallel at $p$).

**Proof.** We assume that $\dim N_1(p) = 2$. we obtain $N_1(p) = \text{Span}\{h_{11}, h_{12}\}$. Moreover $p$ is a minimal point of $M$ i.e.,
\[
h_{11} = -h_{22}.
\]
(4.6)

From (4.5) and (4.6), we have
\[
\langle Dh_{111}, h_{11} \rangle = \langle Dh_{111}, h_{12} \rangle = 0,
\]
\[
\langle Dh_{222}, h_{12} \rangle = 0,
\]
\[ \langle Dh_{222}, h_{11} \rangle = -\langle Dh_{222}, h_{22} \rangle = 0, \]
\[ \langle Dh_{112}, h_{11} \rangle = 0, \]
\[ \langle Dh_{112}, h_{12} \rangle = -\langle Dh_{122}, h_{11} \rangle = \langle Dh_{122}, h_{22} \rangle = 0. \]
Hence we have \( Dh_{111}, Dh_{222}, Dh_{112} \perp N_1(p) \). Since \( \dim T^\perp_p M \leq 3 \) and \( \langle Dh_{111}, Dh_{111} \rangle = \langle Dh_{222}, Dh_{222} \rangle \) in (4.4), we have
\[ Dh_{111} = \pm Dh_{222}. \]
Moreover, from (4.4), we get
\[
\begin{cases}
\langle Dh_{111}, Dh_{112} \rangle = 0, \\
\pm \langle Dh_{111}, Dh_{111} \rangle + 9 \langle Dh_{112}, Dh_{122} \rangle = 0.
\end{cases}
\]
Hence we obtain \( Dh_{111} = 0. \)

We assume that \( \dim N_1(p) = 3. \) We obtain \( T^\perp_p M = N_1(p) = \text{Span}\{h_{11}, h_{12}, \xi\} \) such that \( \langle \xi, \xi \rangle = 1 \) and \( h_{11}, h_{12} \) and \( \xi \) are mutually orthogonal. Since \( \langle Dh_{111}, h_{11} \rangle = \langle Dh_{111}, h_{12} \rangle = 0 \) in (4.4), we have
\[ Dh_{111} = \pm \|Dh_{111}\| \xi. \]
Suppose that \( \|Dh_{111}\| \neq 0. \) Since \( \langle Dh_{111}, Dh_{112} \rangle = \langle Dh_{112}, h_{11} \rangle = 0 \) in (4.4) and (4.5), we have \( Dh_{112} = ah_{12} \ (a \in \mathbb{R}) \). Since \( \langle Dh_{112}, h_{22} \rangle + \langle Dh_{122}, h_{12} \rangle = \langle Dh_{222}, h_{11} \rangle + 3 \langle Dh_{122}, h_{12} \rangle = 0 \) in (4.5) and \( \langle h_{22}, h_{12} \rangle = 0, \) we get
\[ \langle Dh_{222}, h_{11} \rangle = \langle Dh_{122}, h_{12} \rangle = \langle Dh_{112}, Dh_{122} \rangle = 0. \]
Since \( \langle Dh_{111}, Dh_{222} \rangle + 9 \langle Dh_{112}, Dh_{122} \rangle = 0 \) in (4.5), we have
\[ \langle Dh_{222}, Dh_{111} \rangle = \langle Dh_{222}, \xi \rangle = 0. \]
From (4.7), (4.8) and \( \langle Dh_{222}, h_{12} \rangle = 0 \) in (4.5), we have \( Dh_{222} = 0. \) This contradicts the assertion \( \|Dh_{111}\| \neq 0. \) Hence we have \( Dh_{111} = 0. \)

From Proposition 2.3 and Lemma 4.4, we get the following lemma.

**Lemma 4.5.** Let \( n, m, d \) and \( \lambda_1 \) be as in Lemma 4.4. If \( (C_2) \) holds at every point of \( M \), then \( v \equiv 0 \) on \( M \) (i.e., the second fundamental from \( h \) is parallel). Moreover \( \|H\| \) is constant on \( M \) where \( H \) is the mean curvature vector field and
\[ \|H\|^2 = \frac{1}{3} (\Delta + 2(\lambda_1^2 - \lambda_2^2)). \]
Thus the discriminant $\Delta$ is constant on $M$ and the dimension of the first normal space is constant on $M$. Moreover, if the dimension of the first normal space is greater than two, we get

$$\Delta = \frac{1}{4}(\lambda_1^2 - \lambda_1^2 - 3c).$$

**Proof.** Let $U := \{ p \in M \mid \nu(p) > 0 \}$. We shall prove that $U = \emptyset$ ($\emptyset$ is the empty set). Assume that the assertion is false. From Lemma 4.4, we see that $\dim N_1(p) \leq 1$ for every point $p$ of $U$. Hence $U$ is totally umbilic. Then we obtain that the second fundamental form is parallel because of the assumption that $\tilde{M}$ is of constant curvature and $\dim U = 2$. Hence we obtain $\nu(p) = 0$ for every point $p \in U$. This contradicts the assertion that $\nu(p) > 0$ for every point $p \in U$. Hence we have $\nu \equiv 0$ on $M$. Since $M$ is constant isotropic and the second fundamental form is parallel, we obtain that $\|H\|$ is constant on $M$ and the discriminant $\Delta$ is constant on $M$. From Ricci identity, (4.1), (4.2), (4.3) and the fact that $M$ is constant isotropic, we get

$$(D^2h)(x, y, x^2) - (D^2h)(y, x^2) = R^h(x, y)h(x^2) - 2h(R(x, y)x, x)$$

$$= \{2(\lambda_1^2 - \lambda_1^2 + c) - 8\|h(x, y)\|^2\}h(x, y)$$

for every $x, y \in UM$ such that $\langle x, y \rangle = 0$. Since $\nu \equiv 0$ on $M$ and (3.13), we have (4.9).

From Lemma 4.5, we the following theorem.

**Theorem 4.6.** Let $M$ be a two-dimensional connected Riemannian submanifold in an $m$-dimensional Riemannian manifold $\tilde{M}$ of constant curvature $c$ isometrically immersed by $f$ and $m \leq 5$. Let $d = 2$ and $\lambda_1$ be a positive constant. If the condition $(C_2)$ holds for every point of $M$, then the second fundamental form $h$ is parallel on $M$ and $M$ is one of the following:

(a) an extrinsic sphere of constant curvature $c + \lambda_1^2 - \lambda_1^2$,

(b) a non-geodesic minimal submanifold of constant curvature $c/3$ ($> 0, c = 3(\lambda_1^2 - \lambda_1^2)$),

(c) a non-minimal submanifold of constant curvature $(c + \lambda_2^2 - \lambda_2^2)/4$ ($> 0, c \neq 3(\lambda_1^2 - \lambda_1^2), \lambda_1 > \lambda_1$).

If, for every geodesic $\gamma$ in $M$, $f \circ \gamma$ is a helix of order $\tilde{d}$ with curvatures $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{\tilde{d}-1}$ which do not depend on $\gamma$, then $f$ is said to be a *helical immersion of*
order \( \tilde{d} \). Let \( \gamma \) be a geodesic in \( M \) and \( v_1 \) the tangent vector field of \( \gamma \). From (2.1), we have

\[
\begin{align*}
\tilde{\nabla}_v v_1 &= h(v_1^2), \\
\tilde{\nabla}_v h(v_1^2) &= -A_{h(v_1^2)} v_1 + (Dh)(v_1^2).
\end{align*}
\] (4.10)

From (4.10), Proposition 2.3 and Theorem 4.6, we obtain the following fact.

**Corollary 4.7.** Let \( f, M, \tilde{M}, n, m \) and \( \lambda_1 \) be as in Theorem 4.6. Suppose that \( (C_2) \) holds at every point of \( M \). Then \( f \) is a helical immersion of order at most two.

We assume that all conditions of Theorem 4.6 hold. Let \( p \) be a point of \( M \) and \( \sigma \) a circle through \( p \) in \( M \) with the first curvature \( \lambda_1 \) and \( v_1, v_2 \) the Frenet frames of \( \sigma \). Since \( Dh = 0 \), \( M \) is constant isotropic, \( \tilde{\sigma} \) is a 2-regular curve and \( (C_2) \) holds, we see that

\[
\begin{align*}
\tilde{\lambda}_1 \tilde{\omega}_2 &= 3\lambda_1 h(v_1, v_2), \\
\tilde{\lambda}_1 \tilde{\omega}_2 \tilde{\omega}_3 &= -\frac{\tilde{\lambda}_2}{3\lambda_1} (\tilde{\lambda}_1^2 - 3\lambda_1^2) v_2 + (\tilde{\lambda}_2^2 - 3\lambda_1^2) h(v_1^2) + 3\lambda_1^2 h(v_2^2)
\end{align*}
\] (4.11) (4.12)

by Lemma 3.1. Let \( I_\sigma = \{ s \in I \mid \tilde{\omega}_3(s) = 0 \} \) where \( I \) is the domain of \( \sigma \).

If \( I_\sigma \neq \emptyset \), then we have \( \tilde{\lambda}_2 = 0 \) or \( \tilde{\lambda}_2 = \sqrt{2\tilde{\lambda}_1} = \sqrt{6}\lambda_1 \).

In the case where \( \tilde{\lambda}_2 = 0 \), we obtain that \( \tilde{\sigma} \) is a circle. Since \( h(v_1(0), v_2(0)) = 0 \) and \( n = 2 \), we have \( h(x, y) = 0 \) and \( h(x^2) = h(y^2) \) for every \( x, y \in \tilde{U}_pM \) such that \( \langle x, y \rangle = 0 \). Hence we see that \( \tilde{\sigma} \) is a circle through \( p \) with the first curvature \( \lambda_1 \). Then it is clear that the case \( (a) \) of Theorem 4.6 holds.

In the case where \( \tilde{\lambda}_2 = \sqrt{2\tilde{\lambda}_1} = \sqrt{6}\lambda_1 \), from (4.12), we obtain that \( \tilde{\omega}_3 = \sqrt{2}H_\sigma \) where \( H_\sigma = (h(v_1^2) + h(v_2^2))/2 \). Since \( Dh = 0 \) and \( M \) is constant isotropic, we have \( \tilde{\lambda}_3 = ||\tilde{\omega}_3|| \) is constant on \( I \). Hence we have \( \tilde{\lambda}_3 = 0 \), i.e., \( \tilde{\sigma} \) is a helix of order three satisfying that \( \tilde{\lambda}_2 = \sqrt{2} \tilde{\lambda}_1 = \sqrt{6}\lambda_1 \). Since \( h(v_1^2(0)) + h(v_2(0)^2) = 0 \), \( \|h(v_1(0), v_2(0))\| = \|h(v_1^2(0))\| \) and \( n = 2 \), we have \( \|h(x, y)\| = \|h(x^2)\| = \|h(y^2)\| \) for every \( x, y \in \tilde{U}_pM \) such that \( \langle x, y \rangle = 0 \). Hence we see that \( \tilde{\sigma} \) is a helix of order three satisfying that \( \tilde{\lambda}_2 = \sqrt{2} \tilde{\lambda}_1 = \sqrt{6}\lambda_1 \) for every circle \( \sigma \) through \( p \) with the first curvature \( \lambda_1 \). It is clear that the case \( (b) \) of Theorem 4.6 holds.

If \( I_\sigma = \emptyset \), then \( \tilde{\sigma} \) is a 4-regular curve. From (4.11), (4.12) and the fact that \( M \) is constant isotropic, we have

\[
\tilde{\lambda}_3^2 = \frac{\tilde{\lambda}_1^2 \tilde{\lambda}_2^2}{9\lambda_1^2} - \tilde{\lambda}_2^2 + 4\lambda_1^2.
\] (4.13)
From (4.13), we have $\tilde{\lambda}_3$ is constant along $\tilde{\sigma}$. Moreover, from (4.11), (4.12) and (4.13), we get

$$\tilde{\lambda}_1\tilde{\lambda}_2\tilde{\lambda}_3\tilde{V}_4\tilde{u}_4 = -\tilde{\lambda}_2\tilde{\lambda}_3\tilde{w}_2 = \tilde{\lambda}_1\tilde{\lambda}_2\tilde{\lambda}_3(-\tilde{\lambda}_3\tilde{u}_3).$$

From (4.14), we obtain that $\tilde{\sigma}$ is a helix of order four. Then it is clear that the case (c) of Theorem 4.6 holds. Therefore, from Theorem 4.6, we have the following corollary.

**Corollary 4.8.** Let $f, M, \tilde{M}, n, m$ and $\lambda_1$ be as in Theorem 4.6. Suppose that $(C_2)$ holds at every point of $M$. Then $\tilde{\sigma}$ is one of the following:

(a) a circle with the first curvature $\tilde{\lambda}_1$ satisfying $\tilde{\lambda}_1 \geq \lambda_1$ for every circle $\sigma$ with the first curvature $\lambda_1$,

(b) a helix of order three with the first curvature $\tilde{\lambda}_1$ and the second curvature $\tilde{\lambda}_2$ satisfying $\tilde{\lambda}_2 = \sqrt{2}\lambda_1 = \sqrt{6}\lambda_1 = \sqrt{c}(c > 0)$ for every circle $\sigma$ with the first curvature $\lambda_1$,

(c) a helix of order four with the first curvature $\tilde{\lambda}_1$, the second curvature $\tilde{\lambda}_2$ and the third curvature $\tilde{\lambda}_3$ satisfying

$$\tilde{\lambda}_1 > \lambda_1, \quad \tilde{\lambda}_2 = \frac{3\lambda_1\sqrt{c + \tilde{\lambda}_1^2 - \lambda_1^2}}{2\tilde{\lambda}_1^2}, \quad \tilde{\lambda}_3 = \frac{\sqrt{c + \tilde{\lambda}_1^2 - 4\tilde{\lambda}_2^2 + 15\lambda_1^2}}{2}$$

for every circle $\sigma$ with the first curvature $\lambda_1$.

**References**


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