ISOTROPIC MINIMAL SUBMANIFOLDS IN A SPACE FORM

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Let $\tilde{M}^m(\tilde{c})$ be an $m$-dimensional space form of constant curvature $\tilde{c}$, that is, an $m$-dimensional Riemannian manifold of constant curvature $\tilde{c}$. By the Theorem in [5], the author determined $n$-dimensional minimal submanifolds in $\tilde{M}^m(\tilde{c})$ with the sectional curvature not less than $n\tilde{c}/2(n+1)$. We should pay attention to the value next to $n\tilde{c}/2(n+1)$, so that we could classify minimal submanifolds in $\tilde{M}^m(\tilde{c})$ with the sectional curvature not less than it.

In the present paper, we will classify $n$-dimensional isotropic minimal submanifolds in $\tilde{M}^m(\tilde{c})$ with the sectional curvature not less than some value. Indeed, we will prove the following

**Theorem A.** Let $M$ be a connected compact $n$-dimensional ($n \geq 3$) orientable submanifold isotropically and minimally immersed in an $(n+v)$-dimensional space form $\tilde{M}$ of constant curvature $\tilde{c}$. If the sectional curvature of $M$ is not less than $n\tilde{c}/3(n+2)$, then $M$ is of constant curvature $\tilde{c}$ or $n\tilde{c}/3(n+2)$, or the second fundamental form is parallel.

We may assume that $0 < \tilde{c}$ by (2.17) and Remark in §2, that is, $\tilde{M}$ is a sphere $S^n(\tilde{c})$ of constant curvature $\tilde{c}$. When $M$ is of constant curvature, by the results in [2], according as the sectional curvature is $\tilde{c}$ or $n\tilde{c}/3(n+2)$, $M$ is a great sphere of $S^n(\tilde{c})$ or the immersion is the standard minimal one of degree 3 from a sphere into a sphere as stated in [2], which we will call the generalized Veronese submanifold in the present paper. When the second fundamental form is parallel, the above immersion is the planar geodesic one, which is determined in [8]. As a Corollary to Theorem A, using the results in [2], [4] and [8], we have the following

**Theorem B.** Let $M$ be a connected compact $n$-dimensional ($n \geq 3$) orientable submanifold minimally and isotropically immersed in a sphere $S(\tilde{c})$ of constant curvature $\tilde{c}$. If the immersion is full and the sectional curvature $K_\sigma$ satisfies the inequality: $n\tilde{c}/3(n+2) \leq K_\sigma \leq \tilde{c}$, then $M$ is a great sphere of $S(\tilde{c})$, a Veronese submanifold, or a generalized Veronese submanifold.

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§ 1. Preliminaries.

Let \( M \) be an \( n \)-dimensional submanifold immersed in an \((n+p)\)-dimensional space form \( \tilde{M} = \tilde{M}^{n+p}(\varepsilon) \) of constant curvature \( \varepsilon \), (i.e., Riemannian submanifold with induced Riemannian metric). We denote by \( \nabla \) (resp. \( \tilde{\nabla} \)) the covariant differentiation on \( M \) (resp. \( \tilde{M} \)). Then the second fundamental form (the shape operator) \( \sigma \) of the immersion is given by

\[
\sigma(X, Y) = \tilde{\nabla}_XY - \nabla_XY,
\]
where \( X \) and \( Y \) are tangent vectors, and it satisfies \( \sigma(Y, X) = \sigma(X, Y) \). We choose a local field of orthonormal frames \( e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_{n+p} \) in \( \tilde{M} \) in such a way that restricted to \( M \), \( e_1, e_2, \ldots, e_n \) are tangent to \( M \) (and, consequently the remaining vectors are normal to \( M \)). Let \( B \) be the set of all such frames in \( \tilde{M} \). With respect to the frame field of \( \tilde{M} \) chosen above, let \( \tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_{n+p} \) be the field of dual frames. Then the structure equations of \( \tilde{M} \) are given by (*)

\[
d\tilde{\omega}_A = \Sigma \tilde{\omega}_{AB} \otimes \tilde{\omega}_B, \quad \tilde{\omega}_{AB} + \tilde{\omega}_{BA} = 0,
\]

\[
d\tilde{\omega}_A = \Sigma \tilde{\omega}_{AC} \otimes \tilde{\omega}_C - \tilde{\tilde{\omega}}_{AC} \otimes \tilde{\omega}_B.
\]

Restricting these forms to \( M \), we have the structure equations of \( M \):

\[
\omega_a = 0, \quad \omega_i = \Sigma h^a_{ij} \omega_j, \quad h^a_{ij} = h^a_{ji},
\]

\[
d\omega_i = \Sigma \omega_{ij} \otimes \omega_j, \quad \omega_i + \omega_j = 0,
\]

\[
d\omega_{ij} = \Sigma \omega_{ik} \otimes \omega_k - \Omega_{ij}, \quad \Omega_{ij} = \frac{1}{2} \Sigma R_{ijk} \omega_k \otimes \omega_i,
\]

\[
R_{ijk} = \tilde{\tilde{\omega}}_{ik} \otimes \tilde{\omega}_j - \tilde{\tilde{\omega}}_{ij} \otimes \tilde{\omega}_k + \Sigma (h^a_{ik} h^b_{jk} - h^a_{ij} h^b_{jk}).
\]

\[
d\omega_{a\beta} = \Sigma \omega_{a\gamma} \otimes \omega_{\gamma\beta} - \Omega_{a\beta}, \quad \Omega_{a\beta} = \frac{1}{2} \Sigma R_{a\beta} \omega_{a\gamma} \otimes \omega_{\gamma},
\]

\[
R_{a\beta} = \Sigma (h^a_{ik} h^b_{jk} - h^a_{ij} h^b_{jk}).
\]

Then, the second fundamental form \( \sigma \) can be written as

\[
\sigma(X, Y) = \Sigma h^a_{ij} \omega_i \otimes \omega_j \omega_a.
\]

If we define \( h^a_{i_1 \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1}} \) \((1 \leq k)\) by

\[
\sum h^a_{i_1 \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1} \cdots i_{k+1}} \omega_{i_1} \cdots \omega_{i_{k+1}} = d h^a_{i_1 \cdots i_{k+1}} + \Sigma h^a_{i_1 \cdots i_{k+1}} \omega_{i_{k+1}} \omega_i + \Sigma h^a_{i_1 \cdots i_{k+1}} \omega_{i_{k+1}} \omega_i,
\]

then we have

(*) We use the following convention on the range of indices unless otherwise stated: \( A, B, C, \ldots = 1, 2, 3, \ldots, n + p \); \( i, j, k, \ldots = 1, 2, 3, \ldots, n \); \( a, \beta, \gamma, \ldots = n + 1, n + 2, \ldots, n + p \). We agree that repeated indices under a summation sign without indication are summed over the respective ranges.
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(1.5) \[ h_{i',k',l',m'} = \sum h_{i',i} h_{k',k} h_{l',l} h_{m',m} R_{i'j'k'l'} + \sum h_{i',i} R_{j'k'l'm}. \]

If \( M \) is of constant curvature \( c \), then we have

(1.6) \[ R_{i'j'k'l'} = c(\delta_{i'k'}\delta_{j'l'} - \delta_{i'l'}\delta_{j'k'}). \]

The vector \( \sigma(X, X) \) is called the normal curvature vector in the direction of a unit vector \( X \). If the normal curvature vector has the same length \( \lambda \) for any unit tangent vector to \( M \), then the immersion is said to be \( \lambda \)-isotropic. The immersion is called minimal if \( \sum h_{i'm} = 0 \) for all \( \alpha \). We easily see that the immersion is \( \lambda \)-isotropic if and only if the components of the second fundamental form satisfy the following relations ([7]):

(1.7) \[ \sum h_{i'm} h_{i'm} + \sum h_{i'm} h_{j'm} + \sum h_{i'm} h_{k'm} = \lambda^2(\delta_{i'm}\delta_{j'm} + \delta_{i'm}\delta_{j'm} + \delta_{i'm}\delta_{j'm}). \]

Now, we consider the functions on \( M \) defined by

\[ S := \|\sigma\|^2 = \sum h_{i'm} h_{i'm}, \quad L_N := \sum h_{i'm} h_{j'm} h_{k'm} h_{l'm}, \]

\[ K_N := \sum R_{ijkl} R_{ijkl} = \sum (h_{i'm} h_{j'm} - h_{i'm} h_{k'm})(h_{i'm} h_{k'm} - h_{l'm} h_{l'm}). \]

Then we know the following differential equation ([1]):

(1.8) \[ \frac{1}{2} \Delta S = \|\sigma\|^2 + n\varepsilon S - L_N - K_N, \quad \text{where} \quad \|\sigma\|^2 := \sum h_{i'm} h_{i'm}. \]

§ 2. The proof of Theorems.

We suppose that \( M \) is a \( \lambda \)-isotropic and minimal submanifold in a space form \( \tilde{M} \) of constant curvature \( \tilde{c} \). Then, from (1.7), we have

(2.1) \[ \sum h_{i'm} h_{i'm} = \frac{(n+2)}{2} \lambda^2 \delta_{i'i}. \]

that is, \( M \) is an Einstein manifold. If \( n \geq 3 \), as is well known, \( \lambda \) is constant on \( M \). From (1.2) and (1.7) we have

(2.2) \[ 3\sum h_{i'm} h_{i'm} = (\lambda^2 - 2\varepsilon) \delta_{i'i} + (\lambda^2 + \varepsilon)(\delta_{i'k'} \delta_{j'l'} + \delta_{i'k'} \delta_{j'l'} - R_{i'k'l'} - R_{i'k'l'}), \]

which implies

(2.3) \[ \sum h_{i'm} R_{i'k'l'} = (\lambda^2 + \varepsilon) h_{i'm} - \frac{3}{2} \sum h_{i'm} h_{j'm} h_{i'j'}. \]

Since \( \lambda \) is constant, from (1.7), for all \( i, j, k \) and \( l \), we have

\[ \sum h_{i'm} h_{i'm} + \sum h_{j'm} h_{j'm} + \sum h_{k'm} h_{k'm} = -\left( \sum h_{i'm} h_{i'm} + \sum h_{j'm} h_{j'm} + \sum h_{k'm} h_{k'm} \right), \]

which implies

(2.4) \[ \sum h_{i'm} h_{i'm} - \sum h_{j'm} h_{i'm} = \sum h_{j'm} h_{i'm} + \sum h_{i'm} h_{i'm} + 2(\sum h_{i'm} h_{j'm} + \sum h_{j'm} h_{i'm}). \]
Making \((2.4)_{ijmkJ}+(2.4)_{klij}\), we have
\[
\sum h_{ij}^m h_{kj}^m + \sum h_{km}^m h_{ij}^m + \sum h_{km}^m h_{kj}^m = 0.
\]
It follows from \((2.4)_{ijmK}\) and \((2.5)\) that we have
\[
\sum h_{ij}^m h_{km}^m = \sum h_{km}^m h_{ij}^m + \sum h_{km}^m h_{kj}^m.
\]
Since \(M\) is minimal, from \((2.6)\) we get
\[
\sum h_{km}^m h_{kj}^m = 0.
\]
Furthermore, from \((2.6)\) we get
\[
\sum h_{ij}^m h_{km}^m h_{ij}^k h_{km}^k = \sum h_{ij}^m h_{km}^m h_{ij}^k h_{km}^k + 2 \sum h_{km}^m h_{ij}^m h_{ij}^k h_{km}^k,
\]
which imply
\[
3 \sum h_{ij}^m h_{km}^m h_{ij}^k h_{km}^k = \sum h_{ij}^m h_{km}^m h_{ij}^k h_{km}^k.
\]
Letting \(\Delta h_{ij}^m = \sum h_{ij}^m\), from \((2.6)\) we have
\[
\sum h_{ij}^k \Delta h_{ij}^k - \sum (\Delta h_{ij}^k) h_{ij}^k = \sum h_{km}^m h_{ij}^m + \sum h_{km}^m h_{ij}^m,
\]
which implies that
\[
\sum h_{ij}^m h_{km}^m h_{ij}^j = 0.
\]
It follows from \((1.5)\) and \((2.7)\) that we have
\[
\sum h_{ij}^m h_{km}^m h_{ij}^k h_{km}^k = - \sum h_{ij}^m h_{km}^m h_{ij}^k h_{km}^k = - \sum h_{ij}^m h_{km}^m h_{ij}^k h_{km}^k + \sum h_{ij}^m h_{km}^m h_{ij}^k h_{km}^k + \sum h_{ij}^m h_{km}^m h_{ij}^k h_{km}^k,
\]
which, together with \((1.2)\), \((1.3)\), \((1.7)\), \((2.3)\) and \((2.9)\), implies
\[
\sum h_{ij}^m h_{km}^m h_{ij}^k h_{km}^k = \left\{ \frac{n+4}{2} \right\} L_N - n(\bar{\xi} + 2\bar{\lambda}^2)S + 2 \text{Trace}(H^a H^b H^c H^a H^b H^c) + \left\{ \frac{n+10}{2} \right\} L_N - n\bar{\xi} \bar{\lambda} S - \frac{1}{2} \text{Trace} A^2 - 2 \sum \text{Trace} H^a H^b H^c H^a H^b H^c
\]
where \(A = (\sum h_{ij}^m h_{ij}^m)\) and \(H^a = (h_{ij}^m)\). By means of \((1.2)\), \((1.5)\), \((2.1)\) and \((2.3)\), we have
\[
\Delta h_{ij}^m = \sum h_{ij}^m = n(\bar{\xi} - \bar{\lambda}^2) h_{ij}^m - 2 \sum h_{km}^m h_{ij}^m h_{ij}^k.
\]
Since \(S = \sum h_{ij}^m h_{ij}^m = n(n+2)\bar{\lambda}^2/2\) is constant on \(M\), using \((1.7)\), we rewrite \((1.8)\) as
\[
\|\sigma_3\|^2 = 2L_N - n(\bar{\xi} - \bar{\lambda}^2)S.
\]
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Making use of (1.5), (2.7) and (2.12), we have
\[
\begin{align*}
\Sigma h_{ijtm} - \Sigma h_{ijtm} &= \Sigma h_{ipm} R_{pjm} + \Sigma h_{izp} R_{pjm} + \Sigma h_{ijp} R_{jtm} + \Sigma h_{ijm} R_{jtm} \\
\Sigma h_{ijtm} - \Sigma h_{ijtm} &= \Sigma h_{ipm} R_{pjm} + \Sigma h_{izp} R_{pjm} + \Sigma h_{ijp} R_{jtm} + \Sigma h_{ijm} R_{jtm} \\
&+ \Sigma h_{ijp} R_{jtm} h_{jtm} - \Sigma h_{ijm} R_{jtm} h_{jtm}
\end{align*}
\]
\[
\Sigma h_{ijtm} = n(\xi - \lambda^2) h_{ijt} - 2 \Sigma h_{ijm} h_{ijm} - 2 \Sigma h_{jm} h_{jm} - 2 \Sigma h_{jm} h_{jm} - 2 \Sigma h_{jm} h_{jm} \hbox{if } \xi = 0.
\]

Summing up these equations and making use of (2.2)~(2.11), we have
\[
\Delta \| \sigma_{ij} \|^2 = 2 \Sigma h_{ijkm} h_{ijkm} + 2 \Sigma h_{ijkm} h_{ijkm} = \frac{2}{3} \big( h_{ijkm} h_{ijkm} \big) \| \sigma_{ij} \|^2 + 2 \| \sigma_{ij} \|^2 - 6 \Sigma h_{ijm} h_{ijm} h_{ijm} \hbox{if } \xi = 0
\]
that is,
\[
(2.14) \quad \Delta \| \sigma_{ij} \|^2 = 2 \| \sigma_{ij} \|^2 + (2n+3)\xi - 3(n-2)\lambda^2 \| \sigma_{ij} \|^2 - 6 \Sigma h_{ijm} h_{ijm} h_{ijm} \hbox{if } \xi = 0
\]
\[
- 18 \frac{1}{2} \lambda^2 - \lambda^2 \bigg) L_N + 18 n \lambda^2 (\xi - \lambda^2) S
\]
\[+ 9 \text{ Trace } A^2 + 36 \Sigma (\text{Trace } H^a H^b H^c) (\text{Trace } H^a H^b H^c).
\]

where \( \| \sigma_{ij} \|^2 = \Sigma h_{ijkm} h_{ijkm} \). On the other hand, using (2.9) and (2.12), we have
\[
(2.15) \quad \frac{1}{3} \Delta L_N = \frac{3}{4} \Sigma h_{ijm} h_{ijm} h_{ijm} + \Sigma h_{ijm} h_{ijm} h_{ijm} = n(\xi - \lambda^2) L_N - 2 \text{ Trace } A^2.
\]

Now, we consider the function \( f = (2/9) \| \sigma_{ij} \|^2 + (1/4) L_N \). Making use of (2.14) and (2.15), we have
\[
(2.16) \quad \Delta f = \frac{4}{9} \big( \| \sigma_{ij} \|^2 + (2n+3)\xi - 3(n-2)\lambda^2 \big) \| \sigma_{ij} \|^2 + 4 n \lambda^2 (\xi - \lambda^2) S
\]
\[+ \Sigma h_{ijm} h_{ijm} h_{ijm} - 4 \frac{1}{2} \lambda^2 - \lambda^2 \bigg) L_N
\]
\[+ 8 \Sigma (\text{Trace } H^a H^b H^c) (\text{Trace } H^a H^b H^c).
\]

On the other hand, from (1.2) and (2.3) we have
\[
(2.17) \quad \frac{1}{4} \big( \xi - \lambda^2 \big) S - \frac{3}{2} L_N = \Sigma h_{ijm} h_{ijm} R_{kijm} + \Sigma h_{ijm} h_{ijm} R_{kijm}.
\]

For each \( \alpha \), let \( h_{ij}^\alpha, h_{ij}^\alpha, \ldots, h_{ij}^\alpha \) be the eigenvalues of \( H^a \). Then we have
\[
\Sigma_{i,j,m} (h_{ijm}^\alpha h_{ijm} R_{kijm} + h_{ijm}^\alpha R_{kijm}) = \frac{1}{2} \Sigma_{i,j,m} (h_{ij}^\alpha - h_{ij}^\alpha)^2 R_{kijm}
\]
\[
\leq \frac{1}{2} \Sigma_{i,j,m} (h_{ij}^\alpha - h_{ij}^\alpha)^2 c = nc \Sigma_{i,j,m} (h_{ij}^\alpha)^2 = nc \text{ Trace } (H^a)^2,
\]
where \( c \) is the minimum of the sectional curvature of \( M \) and the equality holds only when the sectional curvature is constant. This, together with (2.17), implies
\[ n\left(\bar{\varepsilon} - \frac{\lambda^2}{2}\right)S - \frac{3}{2} L_N \geq ncS, \]

that is,

\[ L_N \geq \frac{n}{3} \{2(\bar{\varepsilon} - c) - \lambda^2\}S, \tag{2.18} \]

where the equality holds only when the sectional curvature is constant.

Premark. It follows from our assumption \( nc/3(n+2) \leq c \) that

\[ 0 \leq \bar{\varepsilon}. \tag{2.19} \]

When \( \tilde{M} \) is simply connected and complete, since \( M \) is a compact minimal submanifold of \( \tilde{M} \), \( \bar{\varepsilon} \) must be positive.

It follows from (1.7) and \( nL_N \geq K_N \) (see [4]) that we have

\[ \frac{n\lambda^2}{n-1} S \leq L_N, \tag{2.20} \]

where the equality holds only when the sectional curvature is constant. From (2.18) and (2.20) we have

\[ 0 \leq L_N - \frac{n\lambda^2}{n-1} S \leq \frac{2n}{9(n+2)} \{2(n+3)\bar{\varepsilon} - \frac{3(n+2)^2}{2(n-1)} \lambda^2\}S. \tag{2.21} \]

Since \( S \) is constant on \( M \), if \( S = \sum h^i_j h^j_i = 0 \), then \( M \) is totally geodesic. Therefore, from now on we may assume that \( S \neq 0 \). Let us now find the lower bounds for \( \sum h^i_j h^j_i h^j_k m h^k_i m, \sum(\text{Trace } H^a H^b H^c)(\text{Trace } H^a H^b H^c) \) and \( \|\sigma_i\|^2 \) when \( S \neq 0 \), that is, we will prove the following.

**Lemma.** If \( S \neq 0 \), then we have the following inequalities;

\[ \|\sigma_i\|^2 \geq \frac{3(n+2)}{(n+4)} \left\{ \frac{n+1}{n} \lambda^2 - \bar{\varepsilon} \right\} \|\sigma_i\|^2 + \frac{6(n+2)}{n(n+4)S} \|\sigma_i\|^2 \left\{ L_N - \frac{n\lambda^2}{n-1} S \right\}, \tag{2.22} \]

\[ \sum h^i_j h^j_i h^j_k m h^k_i m \geq \frac{n\lambda^2}{n-1} \|\sigma_i\|^2 + \frac{1}{nS} \|\sigma_i\|^2 \left\{ L_N - \frac{n\lambda^2}{n-1} S \right\}. \tag{2.23} \]

\[ 2\left\langle \text{Trace } H^a H^b H^c \right\rangle \left\langle \text{Trace } H^a H^b H^c \right\rangle \geq \frac{(n-2)(n+4)}{2(n-1)} \lambda^2 L_N \]

\[ - \frac{(n+4)}{2nS} L_N \left\{ L_N - \frac{n\lambda^2}{n-1} S \right\}. \tag{2.24} \]

**Proof of Lemma.** Taking the length of the tensors

\[ K_{ijkm} = h^i_j h^j_k m - h^i_j h^j_k m + z(\delta_{im} h^i_j + \delta_{jm} h^j_i - \delta_{jk} h^i_m - \delta_{ik} h^j_m) \]

and

\[ L^i_{ijkm} = h^i_j h^j_k m - x(\delta_{ij} h^i_m + \delta_{ik} h^i_m + \delta_{jk} h^j_m) + y(\delta_{jm} h^j_i + \delta_{jm} h^j_m + \delta_{im} h^i_m) \],
where \( z = \| \sigma_3 \|^2 / nS, \) \( x = 2y/(n+2) \) and \( y = (n+2)\| \sigma_3 \|^2 / n(n+4)S, \) we have (2.22) and (2.23), because we have \( \sum (h_{ijkm} - h_{ijkm})h_{ijkm} - h_{ijkm}) = 8\sum h_{ij}h_{jk}h_{ik}h_{km} - 4(\lambda + 4\varepsilon)\| \sigma_3 \|^2 \) from (1.2), (1.3), (1.5), (2.3) and (2.11). Next, taking the length of the tensor

\[
P_{ijkm} = 8\sum h^i_jh^j_m + \sum h^i_jh^j_mh^i_m - a(\delta_{jk}h^m_i + 2\delta_{ik}h^m_j + \delta_{km}h^i_j)
+ b(\delta_{ij}h^m_m + \delta_{km}h^i_j).
\]

we have

\[
(2.25) \quad 2\sum (\text{Trace } H^aH^bH^c)(\text{Trace } H^aH^bH^c) \geq -2(3n+4)a^2S + 2a(3n+8)S - L_N
+ 16abS - 6b^2nS - 4bL_N - (n+2)^2L_N,
\]

because \( \sum h_{ij}h_{m,j}h_{i} = \frac{(n+2)^2}{2} \delta_{ij} \) and \( \sum h_{ij}h_{i}h_{j} = \sum (\sum h_{ij}h_{i}h_{j}) = \sum (\sum h_{ij}h_{j}h_{j}) = \sum \delta_{ij}\delta_{kl} + c(\delta_{ij}\delta_{kl} - \delta_{ij}\delta_{kl} + R_{iklj}) = \lambda^2S - (1/2)L_N \) from (2.2). For some positive constant \( x \) and any positive constant \( \varepsilon \) such that

\[
(2.26) \quad \frac{2x+1}{2(x+1)} - \varepsilon + L_N \leq 2aS < nbS \leq L_N + \varepsilon,
\]

we set

\[
2(3n+4)aS = (3n+8)S - L_N + 4b^2nSx/a + 8(1-x)bS.
\]

Then, from (2.25) we have

\[
(2.27) \quad 2\sum (\text{Trace } H^aH^bH^c)(\text{Trace } H^aH^bH^c) \geq (3n+8)S - L_N - n(2x+1)\varepsilon - (2x+1)nbS + 4(x+1)aS \leq -(2x+1)nbS + 4(x+1)aS
\]

by (2.26), using (2.26), (2.27) implies

\[
(2.28) \quad 2\sum (\text{Trace } H^aH^bH^c)(\text{Trace } H^aH^bH^c) > \frac{(n-2)(n+4)}{2(n-1)} \lambda^2 L_N
\]

\[
- \frac{(n+4)}{2nS} L_N \left( L_N - \frac{n\lambda^2}{n-1} - S \right) + \left( \frac{3n+8}{4(x+1)} \lambda^2 - \frac{n+4}{2nS} \right) \varepsilon.
\]

Since \( \varepsilon \) is any positive constant, we have (2.24).

Now, we will prove Theorem A. Making use of (2.22), (2.23) and (2.24), from (2.16) we have

\[
\Delta f \geq \frac{4(n+1)}{9(n+4)} \left[ 2(n+3)\varepsilon - \frac{3(n+2)^2}{2(n-1)} \lambda^2 \right]
\]

\[
- \frac{3n^2+13n+20}{3n(n+4)S} \left( L_N - \frac{n\lambda^2}{n-1} - S \right) \| \sigma_3 \|^2,
\]

which, together with (2.21), implies
\( \Delta f \geq \frac{2(3n^2+5n-8)}{27(n+4)(n+2)} \left\{ 2(n+3)x - \frac{3(n+2)x}{2(n-1)} \right\} ||\sigma_3|| \geq 0. \)

Therefore, if \( M \) is compact and orientable, we have \( \int_M \Delta f \, d\text{Vol} \, M = 0 \), and so we get \( \Delta f = 0 \) on \( M \), that is,

\[ f \text{ is constant on } M \text{ and } \left\{ 2(n+3)x - \frac{3(n+2)x}{2(n-1)} \right\} ||\sigma_3|| = 0, \]

which implies that if \( M \) is not totally geodesic, then

\[ ||\sigma_3|| = 0 \text{ or } 2(n+3)x = \frac{3(n+2)x}{2(n-1)} x. \]

If \( S \neq 0 \) and \( ||\sigma_3|| \neq 0 \), then it follows from (2.20) and (2.21) that \( M \) is of constant curvature \( \frac{n\varepsilon}{3(n+2)} \). In this case, as stated in [2] or [6], the immersion may be considered as a standard minimal one of degree 3 from a sphere into a sphere. If \( S \neq 0 \) and \( ||\sigma_3|| = 0 \), then the second fundamental form is parallel, and so the immersion is the planar geodesic one which is determined by K. Sakamoto in [8]. Thus we have proved Theorem A.

Next, we will prove Theorem B. If the sectional curvature \( K_\varepsilon \) of \( M \) satisfies the inequality \( \frac{n\varepsilon}{3(n+2)} \leq K_\varepsilon \leq \varepsilon \), then, by Theorem A, we see that \( M \) is of constant curvature \( \varepsilon \) or \( n\varepsilon/3(n+2) \), or the second fundamental form of \( M \) is parallel. Looking over the curvatures of planar geodesic submanifolds in [8], we easily see that \( M \) must be of constant curvature \( \varepsilon \) or \( n\varepsilon/2(n+1) \) when \( ||\sigma_3|| = 0 \). By the results in [5] and [6], according as \( K_\varepsilon \) is \( \varepsilon \), \( n\varepsilon/2(n+1) \), or \( n\varepsilon/3(n+2) \), \( M \) is a great sphere of \( S(\varepsilon) \), a Veronese submanifold, or a generalized Veronese submanifold.

References


Isotropic minimal submanifolds


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