H-SEPARABILITY OF GROUP RINGS
(In memory of Professor Akira Hattori)

By

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Let \( k[G] \) be the group ring of a finite group \( G \) with a coefficient field \( k \). Assume that the characteristic of \( k \) does not divide the order of \( G \). Let \( H \) be a subgroup of \( G \), \( \Delta \) the centralizer of \( k[H] \) in \( k[G] \) and \( D \) the double centralizer of \( k[H] \) in \( k[G] \). The purpose of this paper is to prove that \( k[G] \) is an \( H \)-separable extension of \( D \). For this, a unit in the center \( C \) of \( k[G] \) plays a fundamental role (Lemma 1). Besides, we can prove the well known facts that \( k[G] \) is (finitely generated) projective over \( C \) and \( k[G] \) is a central separable algebra over \( C \), explicitly, by use of this unit.

Denote by \( g_x \) and \( c_x \) the number and the sum of elements in the conjugate class of \( G \) containing the element \( x \) of \( G \), respectively.

**Lemma 1.** \( u=\sum_c (1/g_c)c_x^{c_x} \) is a unit in \( C \).

**Proof.** We first prove that \( (1/g_c)c_x \) and \( c_x^{-1} \) form a dual base of \( C \) over \( k \). Let \( c_x^{c_x} = \sum_c c_x^{a_x} \) where \( a_x \) are integers. This means that each \( z_i (1 \leq k \leq g_x) \) conjugated to \( z \), appears in \( c_x^{c_x} \) \( a_x \) times, that is, for fixed \( k \), the number of pairs \((i,j)\) such that \( y_i x_j = z_i (1 \leq i \leq g_x, 1 \leq j \leq g_x) \) is equal to \( a_x \). So, the number of terms \( x_j^{-1}=z_j^{-1}y_i (1 \leq j \leq g_x) \) is \( a_x \) in \( c_x^{-1} \) and \( c_x^{-1} = \cdots + (a_x g_i/g_x)c_x^{-1} + \cdots \). This proves that \( (1/g_c)c_x^{c_x} \) or equivalently \( (1/g_c)c_x \) and \( c_x^{-1} \) form a dual base of \( C \) over \( k \). Now \( C \) is a separable \( k \)-algebra in the sense of that, for any field extension \( L \) of \( k \), \( C \) is a semisimple \( L \)-algebra. Then \( u=\sum_c (1/g_c)c_x^{c_x} \) is a unit in \( C \) by Theorem 71. 6 in [2] p.482.

Let \( v \) be the inverse of \( u \) in \( C \), \( uv=1 \).

**Corollary 2.** \( \sum_c (1/g_c)c_x \otimes c_x^{-v} \) is a separability idempotent in \( C \otimes_k C \).

**Proof.** It is clear that \( c(\sum(1/g_c)c_x \otimes c_x^{-v})=(\sum(1/g_c)c_x \otimes c_x^{-v})c \) for any \( c \in C \) and \( \Sigma (1/g_c)c_x c_x^{-v}=1 \).

Let \( p \) be the map of \( k[G] \) to \( C \) defined by \( p(a)=(1/n) \sum_{x \in G} xax^{-1} \) for \( a \in k[G] \), where \( n \) is the order of \( G \). The map \( p \) is the projection of \( k[G] \) to \( C \). Then \( p \) is an element of \( \text{Hom}_C(k[G], C) \) which has a left \( k[G] \)-module structure in the usual way.

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**Corollary 3.** \{x \cdot p\} and \{x^{-1}v\} (x \in G) form a projective base of \(k[G]\) over \(C\).

**Proof.** For the identity 1 of \(G\), we have

\[
\sum_{x \in G} (x \cdot p) \cdot x^{-1}v = \sum_{x \in G} p(x) \cdot x^{-1}v = \sum_{x \in G} (1/g_x) c_x x^{-1}v = \sum_{x \in G} (1/g_x) c_x x^{-1}v = 1.
\]

Now, for any \(y \in G\), we have

\[
\sum_{x \in G} (x \cdot p) \cdot y^{-1}v = \sum_{x \in G} p(yx) \cdot x^{-1}v = \sum_{x \in G} p(yx) \cdot x^{-1}v = y.
\]

Now consider the two-sided \(k[G]\)-module \(k[G] \otimes_c k[H]\). Then, for each \(x \in G\), the element \((1/n) \sum_{y \in G} y \otimes xy^{-1}\) defines a two-sided \(k[G]\)-homomorphism of \(k[H]\) to \(k[G] \otimes_c k[H]\). The map \(f_x\) for \(x \in G\), which assigns to \(\sum_{a \in G} a \otimes b_i\) in \(k[G] \otimes_c k[H]\) \(\sum_{a \in G} a \otimes b_i\) in \(k[G]\), is a two-sided \(k[G]\)-homomorphism of \(k[G] \otimes_c k[H]\) to \(k[G]\). Then it is easily verified that \(\sum_{x \in G} f_x\) is the identity map of \(k[G] \otimes_c k[H]\). Thus we have proved the following corollary.

**Corollary 4.** \(k[G] \otimes_c k[H]\) is a two-sided \(k[G]\)-direct summand of the direct sum of \(n\)-copies of \(k[H]\).

If this is the case, then it holds that \(k[G] \otimes_c k[H] \cong \text{Hom}_C(k[G], k[H])\) and \(k[G]\) is \(C\)-finitely generated projective, see [3] p. 112. Therefore \(k[G]\) is a central separable \(C\)-algebra by Theorem 2.1 [1].

Let \(H\) be a subgroup of \(G\) and \(G = \sum_{i=1}^{n} y_i H\) a coset decomposition of \(G\) by \(H\). Denote by \(h_i\) and \(d_i\) the number and the sum of elements in the \(H\)-conjugate class of \(G\) containing the element \(x\) of \(G\), respectively. Let \(\Delta\) be the centralizer of \(k[H]\) in \(k[G]\). Then \(\{d_i\}\) is a \(k\)-basis of \(\Delta\). By the same way as in Lemma 1, it can be verified that \(\{(1/h_i)d_i\}\) and \(\{d_i^{-1}\}\) form a dual basis of \(\Delta\) over \(k\). Let \(q\) be the map of \(\Delta\) to \(C\) defined by \(q(a) = (1/r) \sum_i y_i a y_i^{-1}\), \(a \in \Delta\). It can be shown that \(q\) does not depend on the choice of \(y_i\), and \(q\) is the projection of \(\Delta\) to \(C\).

**Proposition 5.** \(\{(1/h_i)d_i, q\}\) and \(\{d_i^{-1}, q\}\) form a projective base of \(\Delta\) over \(C\).

**Proof.** If we notice that \(q(d_i) = (h_i/q_x)c_x\), the calculation is similar to the proof in Corollary 3 and we shall omit it.

Let \(D\) be the centralizer of \(\Delta\) in \(k[G]\). Then \(D \supset k[H]\) and the centralizer of \(D\) in \(k[G]\) is equal to \(\Delta\).

**Proposition 6.** \(k[G]\) is an \(H\)-separable extension of \(D\).

**Proof.** For a representative \(x\) of an \(H\)-conjugate class of \(G\), define

\[
s_x: k[G] \longrightarrow k[G] \otimes_D k[G] \quad \text{by} \quad s_x(a) = ((1/r) \sum_i y_i \otimes (1/h_i)d_i y_i^{-1})a
\]
and
\[
t_i: k[G] \otimes_D k[G] \rightarrow k[G] \quad \text{by} \quad t_i(\sum a_i \otimes b_i) = \sum a_i d_{i-1} v b_i,
\]
respectively. As \((1/r)\sum_i y_i \otimes (1/h_i) d_i y_i^{-1}\) is in \((k[G] \otimes_D k[G])^H\) and \(d_{i-1} v\) is in \(\Delta\), \(s_i\) and \(t_i\) are two-sided \(k[G]\)-homomorphisms, respectively. If we notice that \(\Sigma d_i (1/h_i) d_i y_i^{-1} d_{i-1} v\) is contained in \(D\), it is easily verified that \(\Sigma s_i t_i\) is the identity map of \(k[G] \otimes_D k[G]\), where the sum is taken over all the \(H\)-conjugate classes of \(G\). Therefore \(k[G] \otimes_D k[G]\) is a two-sided \(k[G]\)-direct summand of a direct sum of finite copies of \(k[G]\) and \(k[G]\) is an \(H\)-separable extension of \(D\).

Even if the characteristic of \(k\) divides the order of \(G\), if the index of \(H\) in \(G\) is a unit in \(k\), \(k[G]\) is always a separable extension of \(k[H]\) by Proposition 3.1 [4]. In this case, it happens that \(k[G]\) may or not be an \(H\)-separable extension of \(D\). Let \(k\) be a field of characteristic two. Take \(G=S_3\) the symmetric group of degree three and \(H=\langle(12)\rangle\). Then \(G=H+(13)H+(23)H\) is a coset decomposition of \(G\) by \(H\). Put \(x_1=(12), x_2=(13)+(23)\) and \(y=(23)+(13)\). Then we have \(\Delta=k[1+kx_1+kx_2+ky]\) and \(D=k[G]^\Delta=D\). The projection \(q\) of \(\Delta\) to \(C\) is given by \(q(a)=(1/3)(1-a-1+(13)a(13)+(23)a(23))\) for \(a \in \Delta\). Then \([q, x_2, q, y\cdot q] \) and \([1+y, x_2, 1]\) form a projective base of \(\Delta\) over \(C\). Define maps \(s_i: k[G] \rightarrow k[G] \otimes_D k[G] (i=1, 2, 3)\) by 
\[
s_1(a)=(1/3)(1 \otimes 1 + (13) \otimes (13) + (23) \otimes (23))a, \quad s_2(a)=(1/3)(1 \otimes x_2 + (13) \otimes x_2 (13) + (23) \otimes x_2 (23))a \quad \text{and} \quad s_3(a)=(1/3)(1 \otimes y + (13) \otimes y (13) + (23) \otimes y (23))a,
\]
respectively. Also define maps \(t_i: k[G] \otimes_D k[G] \rightarrow k[G] (i=1, 2, 3)\) by 
\[
t_1(\sum a_i \otimes b_i) = \sum a_i (1+y) b_i, \quad t_2(\sum a_i \otimes b_i) = \sum a_i x_2 b_i \quad \text{and} \quad t_3(\sum a_i \otimes b_i) = \sum a_i b_i,
\]
respectively. Then \(\sum_{i=1}^3 s_i t_i\) is the identity map of \(k[G] \otimes_D k[G]\) and \(k[G]\) is an \(H\)-separable extension of \(D\). Next, take \(G=S_4\) and \(H=\langle(13), (1234)\rangle\). Then the center \(C\) of \(k[G]\) is a local ring of dimension five over \(k\). On the other hand we can see easily that \(\Delta\) is eight dimensional over \(k\). Therefore \(\Delta\) is not \(C\)-projective and \(k[G]\) is not an \(H\)-separable extension of \(D\).

References


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