FUNDAMENTAL SOLUTION OF CAUCHY PROBLEM
FOR HYPERBOLIC SYSTEMS AND GEVREY CLASS

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§ 1. Introduction

We consider a first order partial differential operator \( L_{t,x} = \frac{\partial}{\partial t} + \sum_{j=1}^{n} A_j(t, x) \frac{\partial}{\partial x_j} + B(t, x) \) in \( \Omega = [0, T] \times \mathbb{R}^n \), whose coefficients are \( m \times m \)-matrices. We call a fundamental solution corresponding to the operator \( L_{t,x} \), a distribution satisfying the following, \( \tau \in (0, T) \), fixed,

\[
\begin{cases}
L_{t,x} K(t, x, \tau, y) = 0, & t \in (0, T) \\
K(\tau, x, \tau, y) = \delta(x-y) I,
\end{cases}
\]

(1.1)

where \( \delta(x) \) denotes the \( n \)-dimensional Dirac distribution and \( I \) the identity matrix. We require that the multiplicity of each characteristic remains constant in a region \( \Omega = [0, T] \times \mathbb{R}^n \) and that the characteristic matrix \( A(t, x, \xi) = \sum A_j(t, x) \xi_j \) is diagonalizable for \( (t, x) \) in \( \Omega \) and \( \xi \) in \( \mathbb{R}^n \setminus 0 \). Moreover we suppose that the coefficients \( A_j(t, x) \) and \( B(t, x) \) are in Gevrey class \( \gamma_s(\mathbb{R})(s \geq 1) \).

Our aim is to construct globally in \( \Omega \) a fundamental solution for the operator \( L_{t,x} \) of this type. When \( T \) is small, Lax [12] treated this problem. In the case of analytic coefficients, Leray [13] and Mizohata [19] analyzed locally a fundamental solution of hyperbolic systems. When \( T \) is large, Ludwig [15] extended the interval of existence for a fundamental solution by use of it's semi-group property. We shall give a more precise expression of a fundamental solution than those of Ludwig. It should be remarked that Duistermaat [3] has recently constructed globally a fundamental solution of the Cauchy problem, applying the theory of Fourier integral operators of Hörmander and Duistermaat [4], [9].

In the first step we shall construct asymptotically a fundamental solution and in the second step we shall obtain successive estimates of it's expansion by use of the method of Mizohata [18], [19] and Hamada [7], [8]. We shall determine the wave front set in Gevrey class of a fundamental solution following the definition of Hörmander [10].

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The work presented here leans heavily on Mizohata's results in [18], and I thank him sincerely.

I announce that we shall construct in the ultra distribution a fundamental solution for non diagonalizable hyperbolic systems in the forthcoming paper.

2. Results

We consider a operator $L_{t,x}=\partial/\partial t + \sum A_j(t,x)\partial/\partial x_j + B(t,x)$ under the following assumptions;

(A.I) each eigen value of $A(t,x,\xi)=\sum A_j(t,x)\xi_j$ is real for $(t,x,\xi)\in \Omega \times \mathbb{R}^n \setminus 0$ and it's multiplicity is constant, that is, $\det(\lambda + A(t,x,\xi)) = \prod_{j=1}^m (\lambda + \lambda_j^{(p)}(t,x,\xi))^{\nu_p}, (\sum \nu_p = m)$, here $\nu_p$ ($p=1\ldots l$) is constant.

(A.II) there exists a positive constant $c_0$ such that

$$\sup_{|t|=1} |\xi^{(p)}(t,x,\xi) - \lambda^{(p)}(t,x,\xi)| \geq c_0$$

(A.III) the characteristic matrix $A(t,x,\xi)$ is diagonalizable.

A function $f \in C^\omega(\Omega)$ is said to be of Gevrey class $\gamma_\omega(\Omega)$ ($s \geq 1$), if there exist constants $C,A$ such that for any $(t,x) \in \Omega$ and for any multi-index $\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_n)$, the following inequality be true;

$$|D^\alpha f(t,x)| \leq CA^{|\alpha|} |\alpha|!$$

here we have set $D^\alpha=(\partial/\partial t)^{\alpha_0}(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$, $|\alpha| = \sum \alpha_i$.

We suppose that the coefficients $A_j(t,x), B(t,x)$ of $L_{t,x}$ are in Gevrey class $\gamma_\omega(\Omega)$. Then all eigen values $\lambda^{(p)}(t,x,\xi)$ are in $\gamma_\omega(\Omega \times \mathbb{R}^n \setminus 0)$.

We denote by $I^{(p)}(t,x,\tau,y,\xi)$ the phase function associated to $\lambda^{(p)}(t,x,\xi)$, that is, a solution satisfying the following non-linear equation;

\begin{align}
\begin{cases}
I_{t}^{(p)} + \lambda^{(p)}(t,x) I_x^{(p)} & = 0 \\
I^{(p)} \big|_{t=0} & = \langle x-y, \xi \rangle,
\end{cases}
\end{align}

here $\langle x, \xi \rangle = \sum_{i=1}^n x_i \xi_i$. To solve this equation, we consider the Hamiltonian system,

$$\begin{cases}
\frac{d}{dt} \tilde{\varphi}^{(p)}(t) = \lambda^{(p)}(t, \tilde{\varphi}^{(p)}, \xi^{(p)}), \\
\frac{d}{dt} \xi^{(p)}(t) = -\lambda^{(p)}(t, \tilde{\varphi}^{(p)}, \xi^{(p)}) \\
\tilde{\varphi}^{(p)}(0) = z, \\
\xi^{(p)}(0) = \xi, (\xi \neq 0).
\end{cases}$$

We write $(\tilde{\varphi}^{(p)}(t), \xi^{(p)}(t))=(\tilde{\varphi}^{(p)}(t,z,\tau,\xi), \xi^{(p)}(t,z,\tau,\xi))$. We can solve globally this system, for $\lambda^{(p)}(t,x,\xi)$ is a homogeneous function in $\xi$. We note that $(\tilde{\varphi}^{(p)}(t), \xi^{(p)}(t))$ is in Gevrey class $\gamma_\omega(\Omega \times \mathbb{R}^n \setminus 0)$ with respect to $(t,z,\xi)$. We put $\tilde{\varphi}^{(p)}(t)=D(\tilde{\varphi}^{(p)}(t))/D(z)$. 

\[\]
Then there exists a positive constant \( \delta > 0 \) such that \( A(t) \neq 0 \) for \( |t| \leq \delta \), because of \( A^{(p)}(\tau) = 1 \). Hence we can solve the equation \( A^{(p)}(t, x, \tau, \xi) = x \) with respect to \( z \) for \( |t-\tau| \leq \delta \). We denote this solution by \( \hat{z}^{(p)}(t, x, \tau, \xi) \). Then we can express the solution of (2.1) as follows,

\[
I^{(p)}(t, x, \tau, y, \xi) = \langle \hat{z}^{(p)}(t, x, \tau, \xi) - y, \xi \rangle.
\]

We note that \( \hat{z}^{(p)}(t, x, \tau, \xi) \) and therefore \( I^{(p)}(t, x, \tau, y, \xi) \) are in \( \gamma([\tau-\delta, \tau+\delta] \times R^n \times R^n \setminus 0) \) with respect to \( (t, x, \xi) \). We denote

\[
I^{(p)}(t, \tau; y) = \bigcup_{(t, x, \xi) \in \gamma([\tau-\delta, \tau+\delta] \times R^n \times R^n \setminus 0)} \{\hat{z}^{(p)}(t, x, \tau, \xi), \hat{z}^{(p)}(t, y, \tau, \xi)\}
\]

Now we analyze the fundamental solution of \( L \). As well known (c.f. [12], [15] and [19]), if \( \delta \) is small, for \( |t-\tau| \leq \delta \) we can express the fundamental solution \( K(t, x, \tau, y) \) as follows,

\[
K(t, x, \tau, y) = \sum_{\beta=1}^{l} K^{(p)}(t, x, \tau, y) + K^{(\alpha)}(t, x, \tau, y),
\]

Here

\[
K^{(p)}(t, x, \tau, y) = \int \exp i I^{(p)}(t, x, \tau, y, \xi)) u^{(p)}(t, x, \tau, \xi) d\xi, \beta = 1, \ldots, l.
\]

Then we obtain

**Theorem 2.1.** Let \( (\tau, y) \) be fixed. For \( |t-\tau| \leq \delta \), we can compute the wave front sets of \( K^{(p)}(t, x, \tau, y) \) in Gevrey class as follows, \( \sigma \geq 1 \),

\[
WF\sigma(K^{(p)}(t, \cdot, \tau, y)) = I^{(p)}(t, \tau; y),
\]

\[
WF_{\sigma-1}(K^{(\alpha)}(t, \cdot, \tau, y)) = \emptyset.
\]

Here the definition of the wave front sets in Gevrey class followed from Hörmander [10].

**Remark.** In the case of analytic coefficients (i.e., \( s=1 \)), the propagation of the analytic wave front sets is studied in [10] and [21]. When \( s>1 \), Friedman [23] showed that the fundamental solution is in \( \gamma_{s-1} \) except the characteristic conoids.

We decompose the interval \((0, T)\) such that \( 0 = t_0 < t_1 < \cdots < t_{d+1} = T, t_j - t_{j-1} = \delta \).

Then it follows from the semi-group property of a fundamental solution that we can write for \( |t-t_j| \leq \delta \),

\[
K(t, x, t_0, y) = K(t, x, t_j, \cdot) K(t_j, \cdot, t_{j-1}, \cdot) \cdots K(t_1, \cdot, t_0, y)
\]

\[
= \sum_{\beta=1}^{l} K_j^{(p)}(t, x, t_0, y) + K_j^{(\alpha)}(t, x, t_0, y),
\]
where we put
\[ K_j^{(p)}(t, x, t_0, y) = K^{(p)}(t, x, t_j, \cdot) K^{(p)}(t_j, \cdot, t_{j-1}, \cdot) \cdots K^{(p)}(t_1, \cdot, t_0, y) \]
for \( j = 1, \ldots, d, |t-t_j| \leq \delta \) and \( p = 1, \ldots, l. \)

**Theorem 2.2.** For \( |t-t_j| \leq \delta \), we have
\[ WF_{j}^p(K_j^{(p)}(t, t_0, y)) = W^{(p)}(t, t_0, y), \quad p = 1, \ldots, l, \]
and
\[ WF_{2s-1}(K_j^{(p)}(t, t_0, y)) = 0, \]
for \( j = 1, 2, \ldots, d. \)

**Remark.** For example, when \( j = 1 \), Theorem 2.2 implies that the singularity of the summation \( \sum_{p=q} K^{(p)}(t, x, t_1, \cdot) K^{(p)}(t_1, \cdot, t_0, y) \) disappears in the Gevrey class \( \gamma_{2s-1}. \)

§ 3. Preliminaries

Let \( \lambda(t, x, \xi) \) be a function in \( \gamma_s(\Omega \times \mathbb{R}^n \setminus 0) \) and homogeneous degree one in \( \xi \). We consider the following equation;
\[ l_t + \lambda(t, x, l_x) = 0, \]
\[ l_{|t=t_0} = \langle x-y, \xi \rangle, \xi \neq 0. \] (3.1)
To solve this nonlinear equation, we consider
\[
\begin{align*}
\lambda(t, x, \xi) &= 0, \\
\frac{d\xi(t)}{dt} &= -\lambda(t, x, \xi), \\
\frac{d\xi(t)}{dt} &= \xi(t) = \xi.
\end{align*}
\]
We write the solution \( (\lambda(t), \xi(t)) = (\lambda(t, x, \xi), \xi(t, z, \tau, \xi)) \).

Then we have,

**Lemma 3.1.** Let \( \tau \) be fixed in \([0, T]\). For \( z \in \mathbb{R}^n \) and \( \xi \in \mathbb{R}^m \setminus 0 \), (3.2) has a unique solution \((\lambda(t), \xi(t))\) which is in \( \gamma_s(\Omega \times \mathbb{R}^n \setminus 0) \) with respect to \((t, z, \xi)\).

Since the Jacobian \( D(\lambda)/D(z) = 1 \) at \( t = \tau \), there exists a positive number \( \delta \) such that \( D(x)/D(z) \neq 0 \) for \( |t - t_0| \leq \delta \). Hence we can solve an equation \( \lambda(t, x, \tau, \xi) = x \) with respect to \( x \) by an implicit function theorem. We denote this by \( \lambda(t, x, \tau, \xi) \). Then we obtain,

**Lemma 3.2.** For \( |t - \tau| \leq \delta \), we can express a solution of (3.1),
\[ l(t, x, \tau, y, \xi) = (\lambda(t, x, \tau, \xi) - y, \xi), \] (3.3)
(3.4) \[ l_2 = \frac{\xi(t, \xi(t, x, \tau, \xi, \tau))}{D(z)}. \]

We denote the Jacobian \( D(\xi(t))/D(z) \) by \( J(t) \). Then we have as well known, (c.f. [5]),

**Lemma 3.3.** For \( |t-\tau| \ll \delta \), we have

\[
\frac{d}{dt} J(t) = J(t) \left[ \sum_{i,j} \frac{\partial}{\partial x_i} (\lambda_i |_{x_i, l_x}) \right]_{x_\tau, l_x}.
\]

here \( l \) is a solution of (3.1)

Let \( A(t, x, \xi) = \sum A_j(t, x) \xi_j \) be a matrix and \( \lambda(t, x, \xi) \) be an eigenvalue of \( A(t, x, \xi) \).

We denote the right eigenvectors and the left eigenvectors by \( h_1, \ldots, h_n \) and \( g_1, \ldots, g_n \), respectively. We write \( H = (h_1, \ldots, h_n) \) and \( G = (g_1, \ldots, g_n) \). Then simple calculations imply

**Lemma 3.4.** For \( j = 1, \ldots, n \), we have

1. \( GA_{t,j}H = \lambda_{t,j}GH, GA_xH = \lambda_{x,j}GH \)
2. \( \sum_{i,j} GA_{t,j}H_{i,j} \xi_{i,j} = \sum_{i,j} \lambda_{t,j}GH_{i,j} \xi_{i,j} + \frac{1}{2} \sum_{i,j} \lambda_{t,j} \lambda_{x,j} G_{i,j} H_{i,j} \) for \( z_{i,j} = z_{j,i} \)
3. \( G_{t,j}A_{x,j}H - G_{x,j}A_{t,j}H = GA_{x,j}H_{t,j} - GA_{t,j}H_{x,j} \)
4. \( GA_{x,j}H_{t,j} - G_{x,j}A_{t,j}H = GH_{t,j} \lambda_{x,j} + GH_{x,j} \lambda_{t,j} \).

§ 4. Asymptotic construction of fundamental solution

We shall construct asymptotically a fundamental solution \( K(t, x, \tau, y) \). We note that the distribution \( \delta(x-y) \) is represented by

\[
\delta(x-y) = \frac{1}{(2\pi)^n} \int \exp i\langle x-y, \xi \rangle d\xi.
\]

Let \( w(t, x, \tau, y, \xi) \) be a function satisfying following equation,

\[
\begin{cases}
L_\tau w(t, x, \tau, y, \xi) = 0 \\
w(t, x, \tau, y, \xi) = \frac{1}{(2\pi)^n} \left\langle \exp i\langle x-y, \xi \rangle \right\rangle I.
\end{cases}
\]

Then we have a fundamental solution \( K(t, x, \tau, y) \) as follows,

\[
K(t, x, \tau, y) = \int_\mathbb{R} w(t, x, \tau, y, \xi) d\xi.
\]
We can construct asymptotically $w(t, x, \tau, y, \xi)$ with respect to $\xi$, provided that the system $L_{t,x}$ satisfies the algebraic conditions (A.I), (A.II) and (A.III) in §2.

We seek $w$ as the following form;

$$w(t, x, \tau, y, \xi) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \left( \exp \left( \sum_{i=1}^{\infty} \beta_{ij} \right) \right) \rho^{-j} w^{(k)}(t, x, \tau, \omega),$$

here

$$l^{(k)}(t, x, \tau, y, \omega) = \langle \xi^{(k)}(t, x, \tau, \omega) - y, \omega \rangle, \omega = \xi/|\xi| \text{ and } \rho = |\xi|.$$

Applying $L_{t,x}$ to $w$, we obtain

$$L_{t,x} [w] = \sum_{j=0}^{\infty} \rho^{-j} \sum_{k=1}^{\infty} \left( \exp \left( \sum_{i=1}^{\infty} \beta_{ij} \right) \right) \rho \left( i l^{(k)}(t, x, \tau, \omega) + A(t, x, f^{(k)}) w^{(k)} + L_{t,x} w^{(k)} \right) = 0.$$

Hence we have

$$(\lambda^{(k)}(t, x, L^{x})) - A(t, x, f^{(k)}) \omega^{(k)} + i L_{t,x} (\omega^{(k)}) = 0 \quad j = 0, 1, 2, \ldots, (\omega^{(k)}) = 0.$$

We put

$$H^{(k)}(t, x, \xi) = (h_{0}^{(k)}(t, x, \xi), \ldots, h_{k}^{(k)}(t, x, \xi)),$$

$$G^{(k)}(t, x, \xi) = (g_{0}^{(k)}, \ldots, g_{k}^{(k)}),$$

here $h_{j}^{(k)}(t, x, \xi)$ (resp. $g_{j}^{(k)}$) is a right (resp. left) eigenvector of $A(t, x, \xi)$ corresponding to $\lambda^{(k)}(t, x, \xi)$.

For $j = 0$, we obtain

$$(\lambda^{(k)}(t, x, L^{x})) - A(t, x, f^{(k)}) \omega^{(k)} + i L_{t,x} (\omega^{(k)}) = 0.$$

Then we obtain as a solution of (4.3)$_{j}$

$$w^{(k)}(t, x, \omega) = \sum_{p=1}^{\infty} H^{(p)}(t, x, L^{x}) \sigma^{(k,p)}(t, x, \omega),$$

where $\sigma^{(k,p)}(t, x, \omega)$ is a $\nu_{k} \times m$ matrix which is determined later on. In general, to solve (4.3)$_{j}$ ($j \geq 1$), it is necessary that

$$G^{(k)}(t, x, f^{(k)}) L_{t,x} (\omega^{(k)}) = 0.$$

We can rewrite (4.5)$_{j-1}$ as an equation of $\sigma^{(p,k)}$, that is,

$$(\lambda^{(k)}(t, x, f^{(k)}) - \lambda^{(p)})^{-1} G^{(p)}(t, x, f^{(p)}) L_{t,x} (\omega^{(k)}) = 0.$$
here we used Lemma 3.4 and $G^{(k)}H^{(k)} = I_{k, (v_k \times v_k)}$-identity matrix),

$$j^{(k)}(t, x, \xi) = G^{(k)}L_iA_iH - \sum_{j=0}^{n} \left( G^{(k)}H^{(k)}(x_{j}) - \frac{1}{2} \lambda_{i,j}^{(k)} G^{(k)}H^{(k)} \right),$$

$$u^{(k)}_j = \sum_{p=0}^{n} H^{(p)}(t, x, \xi) d^{(p)}_j(t, x, \omega).$$

We note that $j^{(k)}$ is invariant under the transformation of variables. For we can rewrite, by virtue of Lemma 3.4, (for simplicity, abbreviating an index $k$),

$$j = \frac{1}{2} \left( G_{i}A_{i}H - G_{i}A_{i}H + GH_{i}A_{i} - GH_{i}A_{i} \right)$$

$$+ 2GH_{i} - \sum \frac{1}{2} GA_{i}A_{i}H + GBH,$$

here we put $L = \xi_{0}I + A_{i}f = \xi_{0}^{2} + \lambda^{(k)}$ and $t = \xi_{0}$. Then we have

$$j = \frac{1}{2} \sum_{j=0}^{n} \left( G_{i}L_{i}x_{j} - G_{i}L_{i}x_{j} \right) I + G(H_{i}f x_{j} - H_{i}f x_{j})$$

$$+ GBH - \frac{1}{2} \sum_{j=0}^{n} GL_{i}A_{i}H.$$

which is evidently invariant under the transformation of variables.

Now we return to the equation (4.8). We transform the variables $x$ into $\tilde{x}^{(k)}(t, z, \tau, \omega)$. Then by use of Lemma 3.3, we can rewrite (4.8) as following,

$$\frac{\partial}{\partial t} + \frac{1}{2} \partial^{(k)}(t) + j^{(k)}(t) \sigma^{(p)}_j(t, \tilde{x}^{(k)}(t), \omega) - i[G^{(k)}L_{i}x(u^{(p)}_{j})]z_{x^{(k)}(i)} = 0$$

We denote by $J^{(k)}(t) = f^{(k)}(t, \tau)$ a solution of the following equation

$$\frac{d}{dt} f^{(k)}(t) = - j^{(k)}(t) f^{(k)}, f^{(k)}(\tau) = I_{k}.$$

We put

$$\sigma^{(p)}_j(t, z, \omega) = J^{(k)}(t)^{1/2} f^{(k)}(t) \sigma^{(p)}_j(t, \tilde{x}^{(k)}(t), \omega).$$

Then we obtain from (4.11)

$$\frac{d}{dt} \sigma^{(p)}_j(t) = i[G^{(k)}L_{i}x(u^{(p)}_{j})]z_{x^{(k)}(i)} = M^{(k)}((u^{(p)}_{j})z_{x^{(k)}(i)}).$$

here $M^{(k)}$ is a first order differential operator in $(t, x)$ and $u^{(p)}_{j}$ is given by (4.10) and (4.7). As an initial condition of (4.12), we obtain from (4.1)

$$\sum_{k=1}^{n} H^{(k)} \sigma^{(k)} = \frac{1}{(2\pi)^{n}} I.$$
and

$$\sum_{k=1}^{l} (\tilde{w}^{j}_k + H^{(k)}a_j^{(k)}) = 0, \quad (j \geq 1)$$

for \( t = \tau \), that is

$$a_j^{(k)}(\tau) = G^{(k)}(\tau, z, \omega)$$

and

$$a_j^{(k)}(\tau) = -G^{(k)}(\tau, z, \omega) \sum_{p=1}^{l} \tilde{w}^{j_p}_p(\tau, z, \omega), \quad (j \geq 1).$$

Summarizing, we have obtained,

$$(4.13)_0\quad \sigma_0^{(k)}(t, z, \omega) = \frac{I}{(2\pi)^n} G^{(k)}(\tau, z, \omega)$$

and for \( j \geq 1 \) and \( k = 1, \cdots, l \),

$$\begin{cases}
\frac{d}{dt} \sigma_j^{(k)}(t) = M^{(k)} \tilde{w}^{j}_j \\
\tilde{w}^{j}_j = N_1^{(k)} \sigma_j^{(k)} + N_2^{(k)} \tilde{w}^{j'}_j \\
\sigma_j^{(k)}(\tau) = G^{(k)}(\tau, z, \omega) \sum_{p=1}^{l} \tilde{w}^{j_p}_p|_{t=\tau}
\end{cases}$$

here \( M^{(k)}, N_1^{(k)} \), and \( N_2^{(k)} \) are first order differential operators in \((t, z)\).

Then we have the following theorem which will be proved in the next section,

**Theorem 4.1.** Let \( \tau \) be fixed in \([0, T]\). For \(|t-\tau| \leq \delta\) and for \( x \in R^n\), we have

$$|D_{x,z}^2 \tilde{w}^{j}_j|_{|x|_1} \leq C_1 A_1^{n+|\beta|+j} (|\alpha| + |\beta|)!j^{j_{2a}-1}$$

and

$$|D_{x,z}^2 \tilde{w}^{j}_j|_{|x|_1} \leq C_1 A_1^{n+|\beta|+j} (|\alpha| + |\beta|)!j^{j_{2a}-1}$$

here \( C_1 \) and \( A_1 \) are positive constants independent of \( \alpha, \beta \) and \( j \).

Therefore we obtain

**Theorem 4.2.** \( \tilde{w}^{j}_j(t, x, \tau, \omega) \) the terms of the expansion \((4.2)\) are homogeneous functions of degree zero with respect to \( \omega \) and are estimated by

$$|D_{x,z}^2 \tilde{w}^{j}_j|_{|x|_1} \leq C_2 A_2^{n+|\beta|+j} (|\alpha| + |\beta|)!j^{j_{2a}-1}, \quad j = 0, 1, 2, \cdots, \quad k = 1, \cdots, l, \quad \text{and for } (t, x) \in [\tau-\delta, \tau+\delta] \times R^n.$$
§ 5. Successive estimate in Gevrey class

We start with a lemma which will be often used in our reasoning (c.f. [6], [18]).

**Lemma 5.1.** Let $p_1$ and $p_2$ be non negative integers and $\alpha = (\alpha_1, \ldots, \alpha_m)$ a multi integer. For any $k > 1$ and $s \geq 1$, we have

\begin{equation}
\sum_{\alpha' + \alpha'' = \alpha} \binom{\alpha}{\alpha'} k^{-|\alpha|} (|\alpha'| + p_1)!^s (|\alpha''| + p_2)!^s \leq \frac{k}{k-1} \left( |\alpha| + p_1 + p_2 \right)!^s \left( \frac{p_1 + p_2}{p_1} \right)^{-1}
\end{equation}

**Proof.** Noting that $\prod_{i=1}^{m} (t+1)^{s_i} = (t+1)^{s_1}$,

we have

\[ \sum_{\alpha' + \alpha'' = \alpha} \binom{\alpha}{\alpha'} = \binom{|\alpha|}{j}, \quad j = 0, 1, \ldots, |\alpha|. \]

In particular for $m=2$,

\[ \left( \frac{\alpha_1}{p_1}, \frac{\alpha_2}{p_2} \right) \leq \left( \frac{\alpha_1 + \alpha_2}{p_1 + p_2} \right) . \]

Hence

\begin{align*}
\sum_{\alpha' + \alpha'' = \alpha} \binom{\alpha}{\alpha'} k^{-|\alpha|} (|\alpha'| + p_1)!^s (|\alpha''| + p_2)!^s \\
\leq \sum_{j=0}^{\left\lfloor \frac{|\alpha|}{2} \right\rfloor} \sum_{\alpha' + \alpha'' = \alpha} \binom{\alpha}{\alpha'} k^{-j} (j + p_1)!^s (|\alpha| - j - p_2)!^s \\
\leq \sum_{j=0}^{\left\lfloor \frac{|\alpha|}{2} \right\rfloor} k^{-j} \binom{|\alpha|}{j} \left( \frac{p_1 + p_2}{j + p_1} \right)^{-1} (|\alpha| + p_1 + p_2)!^s \\
\leq \sum_{j=0}^{\infty} k^{-j} (|\alpha| + p_1 + p_2)!^s \left( \frac{p_1 + p_2}{p_1} \right)^{-1}
\end{align*}

which implies (5.1).

Let $G$ be an open set in $\mathbb{R}^n$ and $\bar{G}$ a closure of $G$.

**Lemma 5.2.** Let $P(x, D) = \sum_{|\beta| = d} a_{\beta}(x) D^\beta$ be a differential operator, $p_1, p_2$ non negative integers and $k$ a positive number $> 1$. Assume

\[ |D^\beta a_{\beta}(x)| \leq C_0 (k^{-1} A)^{|\alpha|} (|\alpha| + p_1)!^s, \quad |\beta| \leq d , \]

\[ |D^\beta u(x)| \leq C A^{|\alpha|} (|\alpha| + p_2)!^s \]
for any multi integer $\alpha$ and for $x \in \mathbb{G}$. Then
\begin{equation}
|D_x^\alpha P(x, D)u(x)| \leq C_0 C \ |m_d| A^{d-\alpha}(\alpha' + p_1 + p_2 + d)!^d
\end{equation}
for $x \in \mathbb{G}$, where $m_d = (m^d-1)(m-1)^{-1}(k-1)^{-1} k$.

**Proof.** Leibniz formula implies
\[
|D_x^\alpha P u| \leq \sum_{|\beta| \leq d} \sum_{\alpha' = \alpha - \beta} \left( \frac{\alpha}{\alpha'} \right) |D_x^{\alpha'} a| |D_x^{\alpha''} u|
\]
which implies (5.2) with (5.1), where we used that
\[
\sum_{|\beta| \leq d} 1 \leq \sum_{j=0}^d m^j = (m^d-1)(m-1)^{-1}.
\]

**Lemma 5.3.** Let $X_j(x, D) = \sum_{i=1}^N a_{ij}(x) \frac{\partial}{\partial x_i} + a_{0j}(x)$, $(j = 1, \ldots, N)$ be first order differential operators. Assume
\[
|D^\alpha a_{ij}(x)| \leq C_0 (k^{-1} A)^{|\alpha|} |\alpha|!, j = 1, \ldots, N, i = 0, \ldots, m,
\]
\[
|D_x^\alpha u(x)| \leq C A^{|\alpha|} (|\alpha| + p)!^d,
\]
for $x \in \mathbb{G}$. Then
\begin{equation}
|D_x^\alpha X_1 X_2 \cdots X_j u| \leq C(C_0 m_1) A^{\alpha' + l} (|\alpha| + l + p)!^d,
\end{equation}
for $x \in \mathbb{G}$ and for $(j_1, \ldots, j_l) \subset (1, \ldots, N)$, where $m_1 = (m+1)(k-1)^{-1} k, k > 1$.

**Proof.** We shall prove our statement by induction with respect to $l$. For $l=1$ it follows from lemma 5.2. In general
\[
|D_x^\alpha X_1(X_2 \cdots X_j u)| \leq \sum_{i=1}^n \sum_{\alpha'} \left( \frac{\alpha}{\alpha'} \right) |D_x^{\alpha'} a_{ij} u| |D_x^{\alpha''} \frac{\partial}{\partial x_i} (X_2 \cdots X_j u)|
\]
\[
+ \sum_{\alpha'} \left( \frac{\alpha}{\alpha'} \right) |D_x^{\alpha'} a_{ij} u| |D_x^{\alpha''} X_2 \cdots X_j u|
\]
\[
\leq C A^{\alpha' + l} C(C_0 m_1)^{-l}(m+1) \sum_{\alpha'} \left( \frac{\alpha}{\alpha'} \right) k^{-|\alpha'|} (|\alpha'| + l + p)!^d
\]
which implies (5.3) with (5.1).

**Lemma 5.4.** Let $G_1$ and $G_2$ be an open set in $\mathbb{R}^{m_1}$ and in $\mathbb{R}^{m_2}$ respectively and $\varphi$ be a mapping from $G_2$ to $G_1$ satisfying
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\[ |D_y^n \varphi(y)| \leq C_0 A_0^{|n|} |\alpha|!^2 \]

for \( y \in \mathcal{G}_2 \). Then for any \( u(x) \) satisfying for \( x \in \mathcal{G}_1 \),

\[ |D_x^m u(x)| \leq CA_0^{|m|}(|\alpha| + p)!^m, \]

if \( A > A_0 \), we have

(5.4) \[ |D_y^n (u \circ \varphi)(y)| \leq C_0 C_0 A_0^{|n|} A_0^{|\alpha|}(|\alpha| + p)!^2 \]

for \( y \in \mathcal{G}_2 \), here \( m_i = (m_i + 1)(k - 1)^{-1} k, k = A/A_0 > 1 \).

**Proof.** Denote \( \varphi(y) = (\varphi_1(y), \ldots, \varphi_{m_1}(y)) \). Then we have

\[
D_y^n (u \circ \varphi)(y) = \sum_{I=1}^{m_1} \frac{\partial \varphi_I}{\partial y_I} \left( \frac{\partial}{\partial x_I} u \right) \varphi(y) \\
= \left( \sum_{I=1}^{m_1} \frac{\partial \varphi_I}{\partial y_I} \frac{\partial}{\partial x_I} + \frac{\partial}{\partial y_I} \right) u(x) \quad |x - y(y)|.
\]

We put

\[ X_I = \sum_{k=1}^{m_1} a_{jk}(y) \frac{\partial}{\partial x_k} + \frac{\partial}{\partial y_I} \varphi_k(y). \]

Nothing that

\[ |D^r a_{jk}(y)| \leq (2C_0 A_0)(k-1)^{|\alpha|} |\alpha|!^r, \quad k = A/A_0 > 1, \]

\[ D_y^n (u \circ \varphi)(y) = (X_1^m, X_2^m, \ldots, X_{m_1}^m) u(x) |x - y(y)|, \]

we obtain (5.4) by virtue of Lemma 5.3.

**Corollary 5.5.** Let \( \varphi \) be given by Lemma 5.4. Then if \( u \in \gamma_4(G_1), u \circ \varphi \in \gamma_4(G_2) \).

**Proof.** It is obvious from (5.4).

Let \( G = (\tau - \delta, \tau + \delta) \times R^{m-1} \) be a band in \( R^m \), \( P(x, D) = \sum_{|j| \leq d-1} a_{j}(x) D^j \) and \( Q(x, D) = \sum_{|j| \leq d-1} b_{j}(x) D^j \), of which coefficients are \( m_1 \times m_2 \) matrices and satisfy

(5.5) \[ |D^r a_{j}(x)| \leq C_0 (k^{-1} A)^{|\alpha|} |\alpha|!^r, |j| \leq d, \]

\[ |D^r b_{j}(x)| \leq C_0 (k^{-1} A)^{|\alpha|} |\alpha|!^r, |j| \leq d-1, \]

for \( x \in \mathcal{G}_1 \), where \( k > 1 \).

We consider the following equations

\[ (5.6) \]

\[ D_{j} F_j = P(x, D) F_{j-1} \quad \text{in } G \]

\[ F_j|_{x_1=\tau} = Q(x, D) F_{j-1}|_{x_1=\tau}, \]

for \( j = 0, 1, 2, \ldots \), where \( F_j(x) \) are \( m_2 \times m_0 \) matrices.
PROPOSITION 5.6. Let \( P(x, D) \) and \( Q(x, D) \) be differential operators of order \( d \) and \( d-1 \) respectively, of which coefficients satisfy (5.5). Assume that \( F_j(x) \) is estimated by

\[
(5.7)_0 \quad |D^r F_j(x)| \leq C A^{|\alpha|} |\alpha|!^l, \ x \in \mathcal{G}.
\]

Then for every \( j \), \( F_j(x) \) satisfying (5.6)_j, can be estimated by

\[
(5.7)_j \quad |D^r F_j(x)| \leq C (C_0 \tilde{m}_d)^{j-1} A^{(|\alpha|-(d-1)j)} \sum_{l=0}^{j-1} \left( \frac{|x_\alpha - \tau| |A|!}{l!} \right) (|\alpha| + j(d-1)+l)!^l
\]

for \( x \in \mathcal{G} \), where \( \tilde{m}_d = (m^{d-1} - 1)(m-1)^{-1}(k-1)^{-1}k, k > 1 \).

PROOF. We shall prove (5.7)_j by induction. For \( j = 0 \) it is trivial. Assume that (5.7)_{j-1} is valid. For \( \alpha = (\alpha_1, \ldots, \alpha_m) = (\alpha, \bar{\alpha}), \alpha_i \neq 0 \), we have from (5.6)_j,

\[
D^r F_j = D^r (D^r PF_{j-1}) = D^r PF_{j-1}, \quad \tau_j = (\alpha, \bar{\alpha}).
\]

Hence

\[
|D^r F_j| \leq \sum_{|\beta| \leq d} \sum_{\alpha' + \alpha'' = \alpha} \left( \frac{\gamma}{\alpha'} \right) |D^r a_j| |D^{r-a'} F_{j-1}|
\]

\[
\leq C_0 \sum_{|\beta| \leq d} \sum_{\alpha'} \left( \frac{\gamma}{\alpha'} \right) k^{-|\alpha'|} A^{|\alpha'|} |\alpha'|!^l C(C_0 \tilde{m}_d)^{j-1}
\]

\[
\times A^{(|\alpha|-(d-1)j)} \sum_{l=0}^{j-1} \left( \frac{|x_\alpha - \tau| |A|!}{l!} \right) (|\alpha'| + (j-1)(d-1)+l + |\beta|)!^l
\]

\[
\leq (C_0 \tilde{m}_d)^{j-1} \left( \frac{k-1}{k} \right) A^{(|\alpha|-(d-1)j)} \sum_{l=0}^{j-1} \left( \frac{|x_\alpha - \tau| |A|!}{l!} \right) \sum_{\alpha'} \left( \frac{\gamma}{\alpha'} \right) k^{-|\alpha'|} |\alpha'|!^l (|\alpha'| + (j-1)(d-1)+l)!^l
\]

\[
+ j(d-1)+l)!^l
\]

which implies (5.7)_j with (5.1).

For \( \alpha = (0, \bar{\alpha}) \), we have

\[
D^r F_j = D^r F_j(\tau, x') + \int_\tau^{x'} (D^r PF_j)(t, x') dt,
\]

here \( x' = (x_s, \ldots, x_m) \). Hence

\[
|D^r F_j(x)| \leq |D^r QF_{j-1}(\tau, x')| + \int_0^{x' - \tau} |D^r PF_{j-1}(t + \tau, x')| dt.
\]

Since it follows from (5.7)_{j-1} that

\[
|D^r F_{j-1}(\tau, x')| \leq C(C_0 \tilde{m}_d)^{j-1} A^{(|\alpha|-(d-1)j)+1} |\alpha| + (j-1)(d-1)!^l,
\]

we obtain by use of Lemma 5.2.
On the other hand, we have by Leibniz’ formula

\[ |D^\rho PF_{j-1}(t+\tau, x')| \leq \sum_{|\alpha| \leq \rho} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) |D^\alpha a(t+\tau, x')| |D^{\beta+\gamma} F_{j-1}(t+\tau, x')| \]

\[ \leq C \left( \frac{k-1}{k} \right) (C_0 m_d)^j A^{\alpha_1, \ldots, \alpha_d} \alpha' \sum_{i=0}^{j-1} \frac{(t A)^i}{i!} \sum_{\alpha'} \left( \begin{array}{c} \alpha' \\ \alpha'' \end{array} \right) k^{-|\alpha'|} |\alpha''| |\alpha'|^{(d-\rho+1)l+1}k! \]

of which integration with respect to \( t \) implies (5.7) with (5.8).

Now we can prove Theorem 4.1 and 4.2. Let \( G = (\tau - \delta, \tau + \delta) \times R^n \times V \), where \( V \) is a neighbourhood of a sphere \( S^{n-1} \). We put in (4.13),

\[
F = \begin{bmatrix}
\sigma_{(1)}^{(1)}, & \cdots, & \sigma_{(1)}^{(d)} \\
W_{(1)}, & \cdots, & W_{(1)}
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
M^{(1)} N_{(1)}, & M^{(1)} N_{(2)}^{(1)}, & \cdots, & M^{(1)} N_{(j)}^{(1)}, & M^{(1)} N_{(d)}^{(1)} \\
D_{(1)} N_{(1)}^{(1)}, & D_{(1)} N_{(2)}^{(1)}, & \cdots, & D_{(1)} N_{(j)}^{(1)}, & D_{(1)} N_{(d)}^{(1)}
\end{bmatrix},
\]

\[
Q = \begin{bmatrix}
G^{(1)} N_{(1)}, & G^{(1)} N_{(2)}^{(1)}, & \cdots, & G^{(1)} N_{(j)}^{(1)}, & G^{(1)} N_{(d)}^{(1)} \\
N_{(1)}, & N_{(1)}^{(1)}, & \cdots, & N_{(j)}^{(1)}, & N_{(d)}^{(1)}
\end{bmatrix}
\]

Then we obtain by virtue of Proposition 5.6 with \( d = 2 \),

\[ |D_{t,s} D_{x} \psi F_{j} | \leq C(C_0 m_2)^j A^{\alpha_1, \ldots, \alpha_d} \left( \frac{(A \psi)^2}{2} \right) \sum_{i=0}^{j} \frac{(A \psi)^i}{i!} |\alpha| + |\beta| + j + l| \]

Noting that

\[ |\alpha| + |\beta| + j + l| \leq 2^{(\alpha_1 + \cdots + \alpha_d) + 1} (|\alpha| + |\beta|) |j + l| \]

we have

\[ |D_{t,s} D_{x} \psi F_{j} | \leq C(4C_0 m_2 \delta A^2)^j (2A)^{\alpha_1 + \cdots + \alpha_d} (|\alpha| + |\beta|) |j + l| \sum_{i=0}^{j} \frac{(j + l)^i}{i!} \delta^{j-l} A^{2(l-j)} \]

\[ \leq C A_{j}^{(\alpha_1 + \cdots + \alpha_d) + 1} (|\alpha| + |\beta|)^{j+1} \]

where

\[ A_j = \max \{ 8^* C_0 m_2 \delta A^2, 2^* A \} . \]

Theorem 4.2 is an immediate result of Theorem 4.1 and Lemma 5.4. For it follows from Lemma 3.1 that the mapping \((t, x, \omega) \mapsto (t, x^{(1)}(t, x, \tau, \omega), \omega)\) is in the class
§ 6. Wave front sets of fundamental solution in Gevrey class

In the term of (4.2), we denote \(|\xi|^{-j}w_{j}^{(\xi)}(t, x, \tau, \omega)\) by \(w_{j}^{(\xi)}(t, x, \tau, \xi)\). Theorem 4.2 implies

\[
|D_{t,x}D_{\xi}^{\beta}w_{j}^{(\xi)}(t, x, \tau, \xi)| \leq C A^{1+|\beta| + p} \left( |\alpha| + |\beta| \right) N^{2d-1} |\xi|^{-j-|\beta|},
\]

for \((t, x) \in [\tau - \delta, \tau + \delta] \times \mathbb{R}^n, \xi \in \mathbb{R}^n \setminus 0, j = 0, 1, \ldots\). Then it follows from the article of Boutet de Monvel and Kree [1] that there exist \(w^{(\xi)}(t, x, \tau, \xi) \in C_{0}^{\infty}([\tau - \delta, \tau + \delta] \times \mathbb{R}^n \times (\mathbb{R}^n \cap |\xi| \geq 1))\) such that

\[
|D_{t,x}D_{\xi}^{\beta} \left( w^{(\xi)}(t, x, \tau, \xi) - \sum_{j=0}^{\infty} w_{j}^{(\xi)}(t, x, \tau, \xi) \right) | \leq C_{1} A^{1+|\beta| + N} \left( |\alpha| + |\beta| \right) N^{2d-1} |\xi|^{-N-|\beta|}
\]

for any positive integer \(N\), \((t, x) \in [\tau - \delta, \tau + \delta] \times \mathbb{R}^n\), and \(\xi \in \mathbb{R}^n, |\xi| \geq 1, p = 1, \ldots, l\).

We define distributions \(W^{(\xi)}(t, x, \tau, y)\) by

\[
W^{(\xi)}(t, x, \tau, y) = \int (\exp i t \xi \cdot \theta(y - x)) \theta(\xi) w^{(\xi)}(t, x, \tau, \xi) d\xi,
\]

where \(\theta(\xi)\) is a \(C^{\infty}\) function in \(\mathbb{R}^n\), which is equal to zero for \(|\xi| \leq 1\) and 1 for \(|\xi| \geq 2\).

In this section our aim is to examine the wave front sets of \(W^{(\xi)}(t, x, \tau, y)\) as a distribution in \(x\) or \((x, y)\).

We shall describe the definition of the wave front sets in Gevrey class, given by Hörmander [10]. We start with

**Lemma 6.1, [10].** Let \(K\) be a compact set in \(\mathbb{R}^n, \varepsilon > 0\) and \(N\) a positive integer. Then there exists a function \(\chi^{\xi}_{\varepsilon}(x) \in C_{0}^{\infty}(\mathbb{R}^n)\) equal to 1 on \(K\) such that \(\text{supp} \chi^{\xi}_{\varepsilon}\) is contained in \(K_{\varepsilon}\) an \(\varepsilon\)-neighborhood of \(K\), and satisfies

\[
|D^{\alpha + \beta} \chi^{\xi}_{\varepsilon}(x)| \leq C_{\alpha} \varepsilon^{-|\beta|} (CN_{\varepsilon}^{-1})^{\beta}, \quad |\beta| \leq N,
\]

where \(C\) depends only on \(n\) and \(C_{\alpha}\) depends only on \(n\) and \(\alpha\).

**Remark.** It follows from Stirling's formula that we have

\[
C_{\alpha}(j + 1)^{j} \leq j! \leq C_{\alpha}(j + 1)^{j}.
\]
Hence, noting that \( N^{|\beta|} |\beta|^{-1} \leq N^n N^{-1}, |\beta| \leq N \), we have

\[
|D^\alpha \chi_\Omega(x)| \leq C A_0 ^n A^{|\alpha|} |\alpha|, |\beta| \leq N.
\]

It follows from Lemma 5.3 that we obtain

**Lemma 6.2.** Let \( X_j = \sum_{i=1}^{n} a_{ji}(x) \frac{\partial}{\partial x_i} + a_{jn}, j=1, \ldots, n \) and \( a_{ji}(x) \) satisfy

\[
|D^\alpha a_{ji}(x)| \leq C_0 (k^{-1} A)^{|\alpha|} |\alpha| \]

for \( x \in K_k, \delta > 1 \). Then we have

\[
|D^\alpha X_j \cdots X_{j_p} \chi_\Omega(x)| \leq C(C_0 n)^p A^{|\alpha|+p} A_0 ^n (|\alpha|+p)!^p
\]

for \( |\alpha|+p \leq N \), where \( n= (n+1)(k-1)^{-1}k \).

**Definition 6.3.** [10]. Let \( x_0 \in \mathbb{R}^n, \xi_0 \in \mathbb{R}^n \setminus 0 \) and \( u \in \mathcal{E}'(\mathbb{R}^n) \). Then we say that \((x_0, \xi_0)\) is in the complement of the wave front sets \( WF(u) \) of \( u \) in the class \( \tau_n \), if there exist a neighborhood \( U \) of \( x_0 \) and a conic neighborhood \( F \) of \( \xi_0 \) such that for \( \xi \in F \)

\[
|\mathcal{E}(\chi_\Omega u(x))| \leq C A_0 ^n N^{|\xi|-N}, N=1, 2, \ldots
\]

are valid for some constants \( \varepsilon, C \) and \( A \) independent of \( N \). Here \( \Omega \) is an \( \varepsilon \)-neighborhood of the closure of \( U \) and \( \mathcal{E} \) stands for the Fourier transform.

We note that we can replace \( \chi_\Omega \) instead of \( \chi_{\Omega'} \). Then the constant \( A \) must be replaced \( A' \) dependent on \( \rho \).

We denote by \( A^{(p)}(t, \tau; y) \) the sets of Hamiltonian flows corresponding to \( \lambda^{(p)}(t, x, \xi) \), that is,

\[
A^{(p)}(t, \tau; y) = \bigcup_{i \in \mathbb{N}, 0} \{ (x^{(p)}(t, y, \tau, \xi), \xi^{(p)}(t, y, \tau, \xi) \}
\]

here \((x^{(p)}, \xi^{(p)})\) is a solution of (3.2) with \( \lambda=\lambda^{(p)}(t, x, \xi), p=1, \ldots, l \).

**Theorem 6.4.** Let \((t, \tau, y)\) be fixed, \( \delta \) a small constant \( > 0 \), and regard \( W^{(p)}(t, x, \tau, y) \) defined in (6.3) as a distribution in \( \mathbb{R}^n \). Then we have

\[
WF_s(W^{(p)}(t, \cdot, \tau, y)) = A^{(p)}(t, \tau; y)
\]

for \( |t-\tau| \leq \delta, p=1, \ldots, l \).

**Proof.** We show at first that

\[
WF_s(W^{(p)}(t, \cdot, \tau, y)) \subset A^{(p)}(t, \tau; y).
\]

Let \((\xi, \xi)\) be not in \( A^{(p)}(t, \tau; y) \). Then there exist a neighborhood \( U \) of \( \xi \) and a conic neighborhood \( F \) of \( \xi \) such that

\[
(\bar{U} \times F) \cap A^{(p)}(t, \tau; y) = \emptyset
\]
for some $\varepsilon > 0$. It is sufficient to prove that

$$
I_{\gamma}(z) = \int (\exp (i u^p(t, x, \tau, \phi, \xi)) \theta(\phi) \chi_{X \geq n} u^{p}(t, x, \tau, \phi, \xi) dxdz
$$

satisfies (6.8) for sufficiently large $|z|, \zeta \in F$. We can write

$$
I_{\gamma}(z) = \rho^a \int (\exp (i \rho t \phi(p) \chi_{X}(x) p(\rho) \phi(p) dxdz)
$$

here, for simplicity we put $\chi_{X} = \chi_{X \geq n}$ and $\phi(p) = \phi(p)(t, x, \tau, \phi, \xi) = \chi_{X} = \xi^{-1}, \rho = |z|$. In order to annihilate the singularity of $w(p)(t, x, \tau, \rho \xi) d \rho d \xi$, with respect to $\xi$, we decompose

$$
I_{\gamma}(z) = \rho^a \int (\exp (i \rho t \phi(p) \chi_{X}(x) \theta(p) \phi(p) + \chi_{X}(1-\theta)u(p) d \xi d\rho)
$$

where $\theta \chi_{X} = \chi_{X \geq n}(z), B = (\rho \geq \rho \xi, \xi \leq \epsilon_1)$. If $\varepsilon$ and $\epsilon_1$ are sufficiently small, $\text{grad}_{X} \phi(p) = I_{\gamma} - \xi$ does not vanish for $\xi \in B$. For, $I_{\gamma}$ is homogeneous degree one in $\xi$ from Lemma 3.2 and $\xi = \xi^{-1} \neq 0$. Hence we may assume that $\phi(p) \neq 0$ for $x \in \tilde{U}$, and $\xi \in B$. Then we obtain from an integration by part, for $\rho \geq \epsilon_1$,

$$
I_{\gamma}(z) = \rho^a \int (\exp (i \rho t \phi(p) \chi_{X}(x) \theta(p) \phi(p) + \chi_{X}(1-\theta)u(p) d \xi d\rho)
$$

Hence it follows from Lemma 6.2 that $I_{\gamma}(z)$ satisfies (6.8). Next we estimate $I_{\gamma}(z)$. It follows from (6.9) that $\text{grad}_{X} \phi(p) \neq 0$ for $x \in \tilde{U}, \zeta \in F$ and $|\zeta| = 1$. Then we can find a first order differential operator $M$ such that $\rho^{-1} M(\exp (i \rho t \phi(p))) = \exp (i \rho t \phi(p))$, that is

$$
M = \sum_{j=1}^{n} \frac{i (\phi(p) \phi(p)) |x|^{-1} + (\phi(p) \phi(p)) |x|^{-1}}{\sum_{j=1}^{n} \left( |x|^{-1} \phi(p) \phi(p) \frac{d}{d x_j} + \phi(p) \phi(p) \frac{d}{d x_j} \right)}
$$

of which coefficients are in Gevrey class $\gamma$ for $x \in \tilde{U}$, and for $|\zeta| \geq \epsilon_1$. Hence we obtain

$$
I_{\gamma}(z) = \rho^{-1} \int (\exp (i \rho t \phi(p) \chi_{X}(x) \theta(p) \phi(p) + \chi_{X}(1-\theta)u(p) d \xi d\rho)
$$

Applying Lemma 6.2, we have for some $C$ and $A$,

$$
|\rho^{-1} M^{-1}(\chi_{X}(1-\theta)u(p)(t, x, \tau, \rho \xi)) \leq C A|\epsilon_1^{-1} N^{-a}(N + N)^{a}
$$

for $x \in \tilde{U}$, and $|\zeta| \geq \epsilon_1$. This implies (6.8) for $I_{\gamma}(z)$. The fact that $A(p)(t, \tau, y) \subset WF_{d}(w(p)(t, \cdot, \tau, y))$ follows from the method of stationary phase. Let $(x, \xi)$ be in $A(p)(t, \tau, y)$, that is, there exists $\xi \in \tilde{U}$, $0$ such that $\tilde{x} = \tilde{x}(p)(t, y, \tau, \xi), \tilde{\xi} = \tilde{\xi}(p)(t, y, \tau, \xi)$. Then it follows from Lemma 3.2 that $\text{grad}_{x} \phi(p) = 0$ for $x = \tilde{x}$ and $\zeta = \tilde{\xi}$. On the other hand, the Hessian of $\phi(p)$ with respect to $(x, \xi)$ (denote by $Q(p)$) is non singular,
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for \(|t-\tau| \leq \delta\), because of \(Q^{(p)}|_{t=\tau} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\). Hence we can apply the method of stationary phase to \(I^{(p)}(\xi)\) (c.f. [3]). It follows,

\[
I^{(p)}(\rho \xi) = (2\pi)^n (\exp -i \rho (x, \xi))|\det(Q^{(p)})|^{-1/2} w^{(p)}(t, x, \tau, \xi) + 0(\rho^{-1}), \rho \rightarrow \infty.
\]

By virtue of (4.4) and (4.13), we have

\[
w^{(p)}(t, x, \tau, \xi) = (2\pi)^n (\exp -i \rho (x, \xi))|\det(Q^{(p)})|^{-1/2} \phi^{(p)}(t, x, \tau, \xi)\psi(t) + 0(\rho^{-1}), \rho \rightarrow \infty.
\]

which does not vanish, because of \(G^{(p)}(\tau, y, \xi)HF^{(p)}(\tau, y, \xi) = 1\). This completes the proof of our theorem.

REMARK. We can regard the integral form (6.3) as the kernel of Fourier integral operator, (c.f. [9]). When \(s=1\), K. Nishiwada [19] investigates the wave front sets of Fourier integral operators in terms of boundary values of holomorphic functions.

As a corollary of Theorem 6.4 we have

**THEOREM 6.5.** Let \((t, \tau)\) be fixed, a small constant \(\delta > 0\), and regard \(W^{(p)}(t, \tau) = W^{(p)}(t, x, \tau, y)\) as a distribution in \(R^{x} \times R_{y}^{n}\). Then for \(|t-\tau| \leq \delta\),

\[
WF(W^{(p)}(t, \tau)) = \bigcup_{(y, \xi) \in R_{x}^{n} \times R^{n}_{\xi} \setminus 0} \{(\xi^{(p)}(t, y, \tau, \xi), y, \xi^{(p)}(t, y, \tau, \xi), -\xi)\}
\]

We next consider the remainder term \(L_{t} W^{(p)}(t, x, \tau, y)\) as a distribution in \(R_{x}^{n}\). It follows evident from Theorem 6.3 that \(WF_{2}(L^{(p)}(t, \tau, y)) \subset A_{L}^{(p)}(t, \tau, y), \) Moreover we can see from the asymptotic expanssion that \(WF_{2}(L^{(p)}(t, \tau, y)) \) is empty. In fact, we can write

\[
R^{(p)}(t, x, \tau, y) = \{(\exp \sum_{k=0}^{N} \lambda^{k} \phi^{(p)}(t, x, \tau, y, \xi)) \phi^{(p)}(t, x, \tau, y, \xi) \}
\]

where

\[
\phi^{(p)}(t, x, \tau, y, \xi) = \{(i(t_{1}^{(p)} + A(t, x, t_{1}^{(p)})) w^{(p)} + L_{t} x w^{(p)} ) \theta(\xi)\}
\]

satisfies

\[
|D_{t} x D_{t} \phi^{(p)}(t, x, \tau, y, \xi)| \leq CA_{1}^{N} N^{1/2} N^{1/2} |\tau|^{N-1} |\xi|^{-N-1}
\]

for \((t, x) \in [\tau-\delta, \tau+\delta] \times R^{n}, \xi \in R_{n}, |\xi| \geq 2,\) and for any positive integer \(N\).

Thus we have proved

**THEOREM 6.6.** Let \(R^{(p)}(t, x, \tau, y)\) be the remainder terms defined by (6.10). Then \(WF_{2}(R^{(p)}(t, \tau, y)) \subset A_{L}^{(p)}(t, \tau, y), \) and \(WF_{2}(R^{(p)}(t, \tau, y)) = \phi \) for \(|t-\tau| \leq \delta, \rho = 1, \ldots, l\).

Now we turn to prove Theorem 2.1. To anihilate the remainder terms \(R^{(p)}(t, x, \tau, y),\) we reduce our problem to an integral equation of Voltera's type, following the method of Kumano-go [11] and Tsutsumi [22].
We denote

\[ W(t, x, \tau, y) = \sum_{p=0}^{l} W^{(p)}(t, x, \tau, y), \]

where \( W^{(p)}(t, x, \tau, \xi) \) is defined by (6.3), \( p = 1, \ldots, l \), and

\[ W^{(0)}(t, x, \tau, \xi) = \int \exp i < x - z, \xi > u^{(0)}(\tau, x, \tau, \xi) d\xi, \]

here

\[ u^{(0)}(t, x, \tau, \xi) = (2\pi)^{-n}(1 - \theta(\xi))I - \left( \sum_{p=1}^{l} u^{(p)}(t, x, \tau, \xi) - (2\pi)^{-n} I \right) \theta(\xi) \]

It follows evidently from (6.2) that

\[ |D_{x}^{\alpha}D_{\tau}^{\beta}W^{(0)}(t, x, \tau, \xi)| \leq C A^{\alpha + |\beta| + N N!|\tau| - N - |\beta|(|\alpha| + |\beta|)) \]

for \( |\xi| \geq 2 \), and therefore

\[ WF_{x=t}(W^{(0)}(t, \cdot, \tau, y)) = \phi, \]

and that we have

\[ W(\tau, x, \tau, y) = \delta(x - y). \]

We shall seek a fundamental solution of \( L_{t, x} \) as the following form

\[ K(t, x, \tau, y) = W(t, x, \tau, y) + \int_{t}^{\tau} \int_{r} W(t, x, \sigma, z) F(\sigma, z, y) dz. \]

Then noting (6.14), we have

\[ L_{t, x}K(t, x, \tau, y) = L_{t, x}W + F(t, x, \tau, y) + \int_{t}^{\tau} \int_{r} (L_{t, x}W)(t, x, \sigma, z) F(\sigma, z, y) dz = 0. \]

Hence we obtain an integral equation

\[ F(t, x, \tau, y) = R(t, x, \tau, y) + \int_{t}^{\tau} \int_{r} R(t, x, \sigma, z) F(\sigma, z, y) dz \]

where we denote

\[ R(t, x, \tau, y) = - L_{t, x}W(t, x, \tau, y) \]

\[ = - \sum_{p=1}^{l} \int \exp i \int_{p}^{(p)}(t, x, \tau, \xi)) r^{(p)}(t, x, \tau, \xi) d\xi - \sum_{p=0}^{l} \int \exp i(x - y, \xi) r^{(p)}(t, x, \tau, \xi) d\xi, \]

where from (6.11) and (6.13) we have
\[ |D_x^\alpha D_{\xi}^\beta \phi^{(P)}(t, x, \tau, \xi)| \leq C_\alpha A_1^{\alpha+|\beta|} |\alpha|^{\alpha-1} |\xi|^{-N-|\beta|} \]

for \(|\xi| \geq 2, p=0, 1, \ldots l\).

**Proposition 6.7.** Let \( R(t, x, \tau, y) \) be the remainder term given by (6.17). There exist positive constants \( C_0 \) and \( A_1 \) such that

\[ |D_x^\alpha D_{\xi}^\beta \phi^{(P)}(t, x, \tau, y)| \leq C_\alpha A_1^{\alpha+|\beta|} |\alpha|^{\alpha-1} |\beta|^{\beta-1} \]

for \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n, |t-\tau| \leq \delta\).

**Proof.** We have

\[ D_x^\alpha D_{\xi}^\beta \phi^{(P)}(t, x, \tau, y) = \sum_{p=0}^l \sum_{|\ell| \geq 1} D_x^\alpha D_{\xi}^\beta \left( \exp \frac{i l(t)}{p} \right)^p \phi^{(P)}(t, x, \tau, y) d\xi. \]

It follows from (6.17) that we have

\[ \left( \frac{1}{|\xi|^2} D_x \right)^\alpha \left( \frac{1}{|\xi|^2} D_{\xi} \right)^\beta \left( \exp \frac{i l(t)}{p} \right)^p \phi^{(P)}(t, x, \tau, y) d\xi \leq C A_1^{\alpha+|\beta|} |\alpha|^{\alpha-1} |\beta|^{\beta-1} |\xi|^{-N-1}, \]

which implies (6.19).

We define

\[ R(t, \tau)u(x) = \int R(t, x, \tau, y)u(y)dy. \]

Then we have

**Proposition 6.8.** Let \( R(t, x, \tau, y) \) be the remainder term, \( A_1 \geq 2A_1 \), given in (6.18) and \( u(x) \) satisfy

\[ |D_x^\alpha u(x)| \leq C_\alpha A_1^{\alpha+|\beta|} |\alpha|^{\alpha-1}, \]

for \( x \in \mathbb{R}^n \). Then there exists a positive constant \( C \) such that

\[ |D_x^\alpha R(t, \tau)u(x)| \leq C A_1^{\alpha+|\beta|} |\alpha|^{\alpha-1}, \]

for \( x \in \mathbb{R}^n, |t-\tau| \leq \delta. \)

**Proof.** We note that

\[ l^{(P)}(t, x, \tau, y, \xi) = \langle \xi^{(P)}(t, x, \tau, \xi), \xi \rangle - \langle y, \xi \rangle \]

\[ = \langle x-y, \xi \rangle + \langle \phi^{(P)}(t, x, \tau, \xi), \xi \rangle \]

Then we have

\[ |D_x^\alpha D_{\xi}^\beta \phi^{(P)}| \leq |t-\tau| C_\alpha A_1^{\alpha+|\beta|} (|\alpha| + |\beta|)^{\alpha} \]

for \( x \in \mathbb{R}^n, |\xi| \geq 1 \). We write
\[
R(t, \tau)u(x) = -\sum_{p=0}^{\infty} \int \left( \exp i \langle x - y, \xi \rangle \right) \psi^{(p)}(y) dy d\xi
\]
\[= -\sum_{p=0}^{\infty} \int \left( \exp i \langle x - y, \xi \rangle \right) \psi^{(p)}(y) dy d\xi.
\]

Hence we have
\[
D_x^n R(t, \tau)u(x) = -\sum_{p=0}^{\infty} \int \left( \sum_{\alpha} \left( \frac{\alpha}{\alpha'} \right) D_x^n \left( \exp i \langle x - y, \xi \rangle \right) D_x^n \left( \psi^{(p)}(y) \right) dy d\xi
\]
\[= -\sum_{p=0}^{\infty} \int \left( \sum_{\alpha} \left( \frac{\alpha}{\alpha'} \right) D_x^n \left( \psi^{(p)}(y) \right) dy d\xi
\]
\[\times \sum_{\alpha} \left( \frac{\alpha}{\alpha'} \right) \left( 1 - \lambda \right)^n D_x^n \left( \psi^{(p)}(y) \right) dy d\xi.
\]

It follows from Lemma 5.4, (6.20) and (6.22) that we have
\[
D_x^n D_t^k u(x) = C_1 A^{(n+1)} \delta(n+1) \Omega \left( \frac{n+1}{2} \right) \Omega \left( \frac{n+1}{2} \right)
\]
where we have put \( k = A_4 / A, \delta = (n+1)(k-1)^{-1} \).

Hence
\[
|D_x^n D_t^k (\psi^{(p)} + y)| \leq C_1 A^{(n+1)} \Omega \left( \frac{n+1}{2} \right) \Omega \left( \frac{n+1}{2} \right)
\]
if \( |t - \tau| \leq \delta \) is sufficiently small, that is,
\[
2^{n-1} C_6 \delta (n+1) k(k-1)^{-1} A \leq 2^{n+1} C_6 \delta (n+1) \leq 1,
\]
here we used \( k \geq 2 \). Moreover we have from (6.18),
\[
|D_x^n D_t^k (\psi^{(p)} + y)| \leq C_0 (k^{-1} A_4)^{n+1} |\alpha|^n |\alpha'| |\alpha''|^n |\xi|^{-n-1}
\]
for \( |\beta| \leq 2n, |\xi| \geq 1, k = A_4 / A \). Hence we obtain from (6.23) and (6.24),
\[
\sum_{\alpha} \left( \frac{\alpha}{\alpha'} \right) \left( 1 - \lambda \right)^n D_x^n \left( \psi^{(p)}(y) \right) dy d\xi
\]
\[\leq C_1 C_2 |\xi|^{-n-1} \sum_{\alpha} \left( \frac{\alpha}{\alpha'} \right) \left( 1 - \lambda \right)^n \Omega \left( \frac{n+1}{2} \right) \Omega \left( \frac{n+1}{2} \right)
\]
\[\leq C_1 C_3 |\xi|^{-n-1} A^{(n+1)} \sum_{\alpha} \left( \frac{\alpha}{\alpha'} \right) \left( 1 - \lambda \right)^n A^{(n+1)} \Omega \left( \frac{n+1}{2} \right) \Omega \left( \frac{n+1}{2} \right)
\]
\[\leq C_1 C_4 |\xi|^{-n-1} A^{(n+1)} \sum_{\alpha} \left( \frac{\alpha}{\alpha'} \right) \left( 1 - \lambda \right)^n A^{(n+1)} \Omega \left( \frac{n+1}{2} \right) \Omega \left( \frac{n+1}{2} \right)
\]
\[\leq C_1 C_4 |\xi|^{-n-1} A^{(n+1)} \Omega \left( \frac{n+1}{2} \right) \Omega \left( \frac{n+1}{2} \right), \text{ (by (5.1))},
\]
which implies (6.21).

Now we shall construct a solution $F$ of the integral equation (6.16). We define inductively

$$F_0(t, x, \tau, y) = R(t, x, \tau, y)$$

$$F_j(t, x, \tau, y) = \int_0^t R(t, x, \sigma, z)F_{j-1}(\sigma, z, \tau, y)dz$$

$$= \int_0^t R(t, \sigma)F_{j-1}(\sigma, \tau)d\sigma.$$ 

Then we can estimate

$$(6.25)_{j} |D_x^{\alpha}D_y^{\beta}F_j(t, x, \tau, y)| \leq C_C C_0 \left| \frac{|t-\tau|}{j!} \right| A_1^{(\alpha - j\beta) |\alpha !|^{\beta - 1} |\beta !^{\beta - 1}}$$

for $|t-\tau| \leq \delta$, $(x, y) \in R^n \times R^n$. (6.25)$_j$ follows from Proposition 6.7. Assume that (6.25)$_{j-1}$ is valid. Then we have from Proposition 6.8

$$|D_x^{\alpha}D_y^{\beta}R(t, \sigma)F_{j-1}(\sigma, \tau)| \leq C_C C_0 \left| \frac{|\sigma-\tau|}{(j-1)!} \right| A_1^{(\alpha - (j-1)\beta) |\alpha !|^{\beta - 1} |\beta !^{\beta - 1}}.$$ 

Integrating this with respect to $\sigma$, we obtain (6.25)$_j$. We define

$$F(t, x, \tau, y) = \sum_{j=0}^{\infty} F_j(t, x, \tau, y)$$

which is a solution of (6.16) and satisfies

$$(6.26) |D_x^{\alpha}D_y^{\beta}F(t, x, \tau, y)| \leq C_C (\exp |t-\tau|C_A)A_1^{(\alpha - j\beta) |\alpha !|^{\beta - 1} |\beta !^{\beta - 1}}.$$

**Proposition 6.9.** Let $W(t, x, \tau, y)$ be given by (6.12), and $u(x)$ satisfied with

$$(6.27) |D_x^{\alpha}u(x)| \leq C_C A_1^{(\alpha !) x} x \in R^n.$$ 

If $s_1 \geq s$, then there exist positive constants $C_a$ and $A_a$ such that

$$(6.28) |D_x^{\alpha}W(t, \tau)u(x)| \leq C_C A_2 A_1^{(\alpha !) x}$$

for $|t-\tau| \leq \delta$, $x \in R^n$.

**Proof.** We have

$$W(t, \tau)u(x) = \sum_{\nu=0}^{\infty} \int_{\nu=0}^{\infty} (\exp i\langle 2^{\nu} - y, \xi \rangle) u^{(\nu)}(t, x, \tau, \xi)u^{(\nu)}(y)dyd\xi$$

$$= \sum_{\nu=0}^{\infty} \int_{\nu=0}^{\infty} (\exp i\langle 2^{\nu} - y, \xi \rangle) u^{(\nu)}(2^{\nu} + y)dyd\xi$$

Hence we have
$$D_x^n W(t, \tau) u(x) = \sum_p \int (\exp i<\eta, \xi>) D_x^n (w^{(p)} u(2^{(p)} + y)) d\xi dy$$

$$= \sum_p \int_{|\eta| \geq 1} (\exp -i<\eta, \xi>)(1+|\eta|^2)^{-n}(1+|\xi|^2)^{-n}(1-J_\eta)^n$$

$$\times (1-J_\xi)^n D_x^n (w^{(p)} u(2^{(p)} + y)) d\xi dy$$

From (6.2), (6.13) and (6.27) we obtain

$$|(1-J_\eta)^n(1-J_\xi)^n D_x^n (w^{(p)} u(2^{(p)} + y))| \leq C_1 C_2 A_n^m |\eta|^m$$

which implies (6.28).

Thus it follows from Proposition 6.8 and 6.9 that we can obtain a fundamental solution such that, if $|t-\tau| \leq \delta$,

$$K(t, x, \tau, y) = W(t, x, \tau, y) + \sum_{j=1}^{\infty} W(t, x, \tau, z) F(\alpha, x, \tau, y) d\alpha d\tau,$$

of which second term belongs to $\gamma_{3\alpha-1}(R_2^1 \times R_2^d)$. Thus we have proved Theorem 2.1.

§ 7. Global construction of fundamental solution

In the previous section we have construct the fundamental solution $K(t, x, \tau, y)$ for $|t-\tau| \leq \delta$, if $\delta$ is sufficiently small. In the present section we shall give an expression of the fundamental solution for any interval $[0, T]$, $T > 0$.

We decompose the interval $[0, T]$ such that $0 = t_0 < t_1 < \cdots < t_{d+1} = T, t_j - t_{j-1} = \delta$. Then it follows from semigroup property that we obtain

$$K(t, x, t_0, y) = K(t, x, t_j, \cdot) K(t_j, \cdot, t_{j-1}, \cdot) \cdots K(t_1, \cdot, t_0, y)$$

for $|t-t_j| \leq \delta$. We put

$$K_j^{(p)}(t, x, t_0, y) = W^{(p)}(t, x, t_j, \cdot) W^{(p)}(t_j, \cdot, t_{j-1}, \cdot) \cdots W^{(p)}(t_1, \cdot, t_0, y)$$

for $|t-t_j| \leq \delta, j = 0, 1, \cdots, d$, and $p = 1, \cdots, l$, where $W^{(p)}(t, x, \tau, y)$ is given by (6.3) for $|t-\tau| \leq \delta$. Then we can express

$$K(t, x, t_0, y) = \sum_{p=1}^d K_j^{(p)}(t, x, t_0, y) + K_j^{(0)}(t, x, t_0, y)$$

for $|t-t_j| \leq \delta$. Our purpose is to prove that

$$WF(K_j^{(p)}(t, \cdot, t_0, y)) = A^{(p)}(t, t_0; y), p = 1, \cdots, l,$$

$$WF_{2d-1}(K_j^{(0)}(t, \cdot, t_0, y)) = \phi$$

for $|t-t_j| \leq \delta$. $j = 0, \cdots, d$. 
We define $A^{(p)}(t, \tau)$ by
\[
A^{(p)}(t, \tau)(y, \xi) = (\xi^{(p)}(t, y, \tau, \xi), \xi^{(p)}(t, y, \tau, \xi)).
\]
Let $F$ be a set in $\mathbb{R}^n \times \mathbb{R}^n \setminus 0$. We write
\[
A^{(p)}(t, \tau) F = \{\xi^{(p)}(t, y, \tau, \xi), \xi^{(p)}(t, y, \tau, \xi); (y, \xi) \in F\},
\]
where $(\xi^{(p)}, \xi^{(p)})$ is a solution of (3.2) with $\lambda = \lambda^{(p)}, p = 1, \ldots, I$. Then we have
\[
A^{(p)}(t, \tau), A^{(p)}(\tau, \sigma) = A^{(p)}(t, \sigma)
\]
\[
A^{(p)}(t, \tau), A^{(p)}(\tau, t) = I
\]
for any $(t, \tau, \sigma)$.

**THEOREM 7.1.** Let $u$ be in $S'(\mathbb{R}^n)$ and $s' \geq s$. Then
\[
WF_s(W^{(p)}(t, \tau) u) \subset A^{(p)}(t, \tau) WF_s(u).
\]
for $|t - \tau| \leq \delta, p = 1, \ldots, I$.

Since $W^{(p)}(t, x, \tau, y)$ is in $S'(\mathbb{R}^n)$ with respect to $x$ for $|t - \tau| \leq \delta$, we obtain

**COROLLARY 7.2.**
\[
WF_s(K_{j, p}^{(p)}(t, \cdot, t_0, y)) \subset A^{(p)}(t, t_0; y),
\]
for $|t - t_0| \leq \delta, j = 0, 1, \ldots, d, p = 1, \ldots, I$.

**PROOF OF THEOREM 7.1.** Let $K$ be a neighborhood of $x_0$ and $\chi_N(x) = \chi_N(x)$. Put
\[
I_N(\zeta, y) = \int (\exp -i(x, \xi)) \chi_N(x) \nabla^{(p)}(t, x, \tau, y) dx
\]
\[
= \int (\exp (-i(x, \zeta) + i(x^{(p)}(t, x, \tau, \xi) - y, \xi))) \chi_N(x) \nabla^{(p)}(t, x, \tau, \xi) d\xi dx.
\]
Then there exists a positive constant $r$ such that for any positive integer $m$ and for $|y| \geq r$
\[
(7.4) \quad \sum_{|\alpha| \leq m} |D^\alpha I_N(\zeta, y)| \leq c_m (1 + |y|)^{-m} A^N |\zeta|^{-N} N_1, N = 1, 2, \ldots,
\]
where $c_m$ depends only on $m$. For, $\nabla \chi_N(x) = \chi_N(x) - y \neq 0$ and $\chi \in \text{supp} \chi_N(x)$, if $r$ is sufficiently large. Let $\chi_N(y) = \chi_N(y)$, where $B_r = \{y, |y| \leq 2r\}$. Then we have
\[
\mathcal{F}(\chi_N(x) W^{(p)}(t, x, \tau, y))(\zeta) = \langle I_N(\zeta, \cdot), \chi_N u \rangle + \langle I_N(\zeta, \cdot), (1 - \chi_N) u \rangle.
\]
Then the second term can be estimated by $c_m |\zeta|^{-m} A^N N^1 \|u\|$ by use of (7.4), where $m$ is the order of the distribution $u$. Let $K_1$ be the intersection of $B_r$ and a neighbor-
hood of the projection of $WF(u)$ into $R^2$ and $\gamma(y) = \gamma(y).$ Then we have

$$|\mathcal{F}((1-\gamma(y))\chi(u)(\xi))| \leq C|\xi|^{-N}A^N \gamma N. \quad N = 1, 2, \ldots$$

for any $\xi \neq 0.$ Hence we have

$$|\langle I_\mathcal{N}(\zeta, \cdot), (1-\gamma(y))\chi(u) \rangle| \leq C|\xi|^{-N}A^N \gamma N.$$  

Moreover for $(y, \xi) \in WF(u), y \in \text{supp} \chi$, we have

$$|\mathcal{F}(\gamma(y)\chi(u)(\xi))| \leq C|\xi|^{-N}A^N \gamma N.$$  

and for $(y, \xi) \in WF(u)$ and $(x, \zeta/|\eta|) \in \mathcal{F}^{(p)}(t, \epsilon)WF(u),$

$$d_{x, \zeta}(|\langle \gamma(y) - y, \xi - \langle x, \zeta^p/|\eta|^p \rangle \rangle| \neq 0.$$  

Hence we obtain

$$|\langle I_\mathcal{N}(\zeta, \cdot), \gamma(y)\chi(u) \rangle| \leq C|\xi|^{-N}A^N \gamma N.$$  

Thus we have proved our theorem.

Denote by $WF(u)$ the wave front sets with respect to $C^\infty$ functions. Then it holds that (c.f. [10]),

$$WF(u) \subseteq WF(u).$$  

Hence to prove that

$$WF(K^{(p)}(t, \cdot, t_0, y)) \supseteq \mathcal{F}^{(p)}(t, t_0, y)$$  

it suffices to indicate

(7.4)  

$$WF(K^{(p)}(t, \cdot, t_0, y)) \supseteq \mathcal{F}^{(p)}(t, t_0, y)$$  

for $|t-t_0| \leq \delta, j=1, \ldots, d, p=1, \ldots, l.$

**Lemma 7.3.** [3]. Let $u$ be in $\mathcal{D}'(R^n).$ Then $(x_0, \xi_0) \not\in WF(u)$ if and only if for any real valued $C^\infty$ function $\varphi(x)$ with $d_\varphi(x_0) = \xi_0$ there exists an open neighborhood $U_0$ of $x_0$ such that for any $\gamma(x) \in C^\infty_0(U_0)$ we have

$$\langle (\exp - i\rho \varphi)\chi, u \rangle = 0(\rho^{-N}) \quad \text{for } \rho \to \infty$$  

uniformly with respect to $\varphi.$

We can express

$$K^{(p)}(t, x, t_0, y) = \int (\exp - i\rho \varphi(t, x, y, t_0, \theta))a^{(p)}(t, x, \theta)d\theta,$$

where

$$\theta = (\xi(x), y^*(x), \xi(x-1), y^*(x-1), \ldots, y^{(x)}, \xi(x)) \in R^{(x+1)n},$$

$$\varphi(t, x, t_0, \theta, y) = \langle \xi^{p}(t, x, t, \xi(x)) - y^{p}(t, x, \xi(x)) \rangle$$
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\[ + \sum_{k=2}^{J} \langle \mathcal{Z}^{(p)}(t_k, y^{(k)}, t_{k-1}, \xi^{(k-1)}) - y^{(k-1)}, \xi^{(k-1)} \rangle \]

\[ + \langle \mathcal{Z}^{(p)}(t_1, y^{(1)}), t_0, \xi^{(0)} \rangle \]

and \( a_j^{(p)}(t, x, t_0, \theta) = \mathcal{U}^{(p)}(t, x, t_j, \xi^{(j)})\mathcal{U}^{(p)}(t_j, y^{(j)}, t_{j-1}, \xi^{(j-1)}) \cdots \mathcal{U}^{(p)}(t_1, y^{(1)}, t_0, \xi^{(0)}) \). It is obvious that \((x, d_{xj}^{(p)}) \in \mathcal{A}^{(p)}(t, t_0; y) \) if and only if \( d_{xj}^{(p)} = 0 \). Hence we have

\[ (7.5) \quad \mathcal{A}^{(p)}(t, t_0; y) = \{(x, d_{xj}^{(p)}); \quad d_{xj}^{(p)} = 0 \} \]

We note that the rank of \( d_{xj}^{(p)}d_{xj}^{(p)} = (2j + 1)n \). For,

\[ d_{xj}^{(p)}d_{xj}^{(p)} = \begin{pmatrix} A_{j-1} & 0 & \cdots & 0 \\ 0 & A_{j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{1} \end{pmatrix} \]

where \( A_{j-1} = d_{xj}^{(p)}(t, x, t_j, \xi^{(j)}), \quad A_k = d_{xk}^{(p)}d_{xk}^{(p)}(t_k, y^{(k)}, t_{k-1}, \xi^{(k-1)}) \) and \( I \) the \( n \times n \) identity matrix, all are nonsingular.

We shall prove (7.4) by use of the method of stationary phase. To do so, we need

**Lemma 7.4.** [3] Let \( \phi(x) \) be a real valued function with \((\bar{x}, \bar{\xi}) \in \mathcal{I}^{(p)}(t, t_0; y), \bar{\xi} = d_{x}\phi(\bar{x}) \). Then the matrix \( d_{xj}^{(p)}(\phi_j^{(p)} - \phi) \) is nonsingular at \((\bar{x}, \bar{\xi})\), if and only if

(i) the rank of \( d_{xj}^{(p)}d_{xj}^{(p)} = (2j + 1)n \)

(ii) the graph \((x, d_{xj}^{(p)}) \) and \((x, d_{xj}^{(p)}), d_{xj}^{(p)} = 0 \) intersect transversally at \((x, \xi)\).

**Lemma 7.5.** Let \((\bar{x}, \bar{\xi}) \in \mathcal{I}^{(p)}(t, t_0; y) \). There exists a non symmetric matrix \( R \) such that \( \phi(x) = \langle x - \bar{x}, \bar{\xi} \rangle + 1/2 \langle R(x - \bar{x}), x - \bar{x} \rangle \) and \( d_{xj}^{(p)}(\phi_j^{(p)} - \phi) \) is nonsingular when \( d_{xj,0}(\phi_j^{(p)} - \phi) = 0 \).

**Proof.** It follows from Lemma 7.4 and (7.5) that \( d_{xj}^{(p)}(\phi_j^{(p)} - \phi) \) is nonsingular if and only if the graph \((x, d_{xj}) \) and \( \mathcal{A}^{(p)}(t, t_0; y) \) intersect transversally at \((\bar{x}, \bar{\xi})\). The transversality means that

\[ T_{\bar{x}, \bar{\xi}} \mathcal{G}(x, d_{xj}) \cap T_{\bar{x}, \bar{\xi}}(\mathcal{A}^{(p)}(t, t_0; y)) = \{0\} \]

where

\[ T_{\bar{x}, \bar{\xi}} \mathcal{G}(x, d_{xj}) = \{ (\partial_x, R\partial_x); \partial_x \in R^n \} \]
Hence the transversality implies that $Rd_{x}(x^{(p)})-d_{t}x^{(p)}$ is non singular. Since the rank of $(x^{(p)}, \xi^{(p)})$ is equal to $n$, we can find $R$ such that $\det(Rd_{x}(x^{(p)})-d_{t}x^{(p)}) \neq 0$.

Now we prove (7.4). Denote by $Q_{j}^{(p)}$ the matrix $d_{t}(x, y)(\phi_{j}^{(p)}-\phi)$. Let $(\dot{x}, \dot{\xi}) = (x^{(p)}(t, y, t_{0}, \omega), \xi^{(p)}(t, y, t_{0}, \omega))$ and $\chi(x) \in C^{\infty}$, its support contained in a neighborhood of $\dot{x}$. Then by virtue of the method of stationary phase, we obtain

$$\langle \exp(-i\rho \phi)\chi, K_{j}^{(p)}(t, \cdot, t_{0}, y) \rangle$$

$$= \int \int \left\{ \exp i(\phi_{j}^{(p)}(t, x, t_{0}, \theta, y)-\rho \phi(x)) \right\} \chi(x) \alpha_{j}^{(p)}(t, x, t_{0}, \theta) dxd\theta$$

$$= \rho^{(j+1)n} \int \int \left\{ \exp i(\phi_{j}^{(p)}-\rho \phi) \chi(x) \alpha_{j}^{(p)}(t, x, t_{0}, \rho \xi^{(p)}, y^{(p)}, \cdots, \rho \xi^{(p)}) dxd\theta \right\}$$

$$= \rho^{(j+1)n} \left\{ \left( \frac{2\pi}{\rho} \right)^{(j+1)n/2} \left| \det Q_{j}^{(p)} \right|^{-1/2} \left( \exp i(\pi/4) \text{sgn} Q_{j}^{(p)} \alpha_{j}^{(p)} \right) \right\} d_{(x, y)}(\phi_{j}^{(p)}-\phi) = 0$$

$$+ O(\rho^{-1}).$$

For $(x, \theta)$ such that $d_{t}(x, \theta)(\phi_{j}^{(p)}-\phi)=0$, that is, $(x, \phi_{j}(y, \omega), (y^{(k)})=A^{(p)}(t, t_{0})(y, \omega))$ and $\xi^{(p)}(\omega)$, we have from (4.4) and (4.13),

$$\alpha_{j}^{(p)} = \left\{ \exp i(\pi/4) \text{sgn} Q_{j}^{(p)} \right\} \left| \det Q_{j}^{(p)} \int \int \left| A^{(p)}(t, t_{0}) \right|^{-1/2}$$

$$\times H^{(p)}(t, \dot{x}, \dot{\xi}) J^{(p)}(t, t_{0}) G^{(p)}(t_{0}, \omega) \left( \frac{1}{2\pi} \right)^{(j+1)n}$$

$$\neq 0.$$

Hence $(\dot{x}, \dot{\xi}) \in WF(K_{j}^{(p)}(t, \cdot, t_{0}, y))$. Thus we have proved (7.2).

**Lemma 7.6.** Let $y$ be fixed in $R^{n}$ and $\delta>0$, small. Then for $p \neq q$ and $0<|\sigma-\tau|<\delta$, we have

(7.6) \quad $A^{(p)}(\sigma, \tau; y) \cap A^{(q)}(\sigma, \tau; y) = \phi$

and

(7.7) \quad $A^{(p)}(\sigma, \tau) A^{(q)}(\sigma, \tau; y) \cap \{(y, R^{n}\setminus 0) = \phi \}$.

**Proof.** Let $(\dot{x}, \dot{\xi})$ be in $A^{(p)}(\sigma, \tau; y) \cap A^{(q)}(\sigma, \tau, y)$, that is, $\dot{x} = x^{(p)}(\sigma, y, \tau, \omega) = x^{(q)}(\sigma, y, \tau, \eta)$ and $\dot{\xi} = \xi^{(p)}(\sigma, y, \tau, \omega) = \xi^{(q)}(\sigma, y, \tau, \eta)$. On the other hand we have

$$\frac{d}{dt} \dot{x}^{(p)} = \lambda_{t}^{(p)} \dot{x}^{(p)}(t, \dot{x}^{(p)}(t), \dot{\xi}^{(p)}(t))$$

$$= \lambda_{t}^{(p)}(\sigma, \dot{x}, \dot{\xi}) + 0(t-\sigma)$$
Hence \( \mathcal{E}^{(p)}(\sigma) \rightarrow y = \lambda^{(p)}(\sigma, \xi) \xi(\sigma - \tau) + O(\sigma - \tau)^2 \). Similarly we have

\[
\mathcal{E}^{(q)}(\sigma) \rightarrow y = \lambda^{(q)}(\sigma, \xi) \xi(\sigma - \tau) + O(\sigma - \tau)^2
\]

Since \( \lambda^{(q)}(\sigma, \xi) \neq \lambda^{(p)}(\sigma, \xi) \), we have \( \mathcal{E}^{(p)}(\sigma, y, \tau, \omega) \neq \mathcal{E}^{(q)}(\sigma, y, \tau, \eta) \) for \( 0 < |\sigma - \tau| \leq \delta \), if \( \delta \) is small. This is contradiction. Put \( \xi^{(q)}(\tau) = \xi^{(q)}(\tau, y, \sigma, \omega) \) and \( \xi^{(q)}(\tau) = \xi^{(q)}(\tau, y, \sigma, \omega) \). Then we have

\[
\mathcal{E}^{(p)}(\sigma, \xi^{(q)}(\tau), \tau, \xi^{(q)}(\tau)) - y
\]

\[
= \mathcal{E}^{(p)}(\sigma, \xi^{(q)}(\tau), \tau, \xi^{(q)}(\tau)) - \mathcal{E}^{(q)}(\tau) + \mathcal{E}^{(q)}(\tau) - y
\]

\[
= (\sigma - \tau) \lambda^{(p)}(\tau, \xi^{(q)}(\tau)) + (\sigma - \lambda^{(q)}(\sigma, y, \omega) + O(\sigma - \tau)^2
\]

\[
\neq 0,
\]

for \( 0 < |\sigma - \tau| \leq \delta \), if \( \delta \) is small. Thus we have proved (7.7).

**Proposition 7.7.** ([14], [17]). Let \( u_0 \) be is \( \gamma_{2t-1}(R^n) \) and \( f(t, x) \) be is \( \gamma_{2t-1} \) with respect to \( x \) and continuous with respect to \( t \). Then a solution of following equation is in \( \gamma_{2t-1}(R^n) \) with respect to \( x \),

\[
\begin{align*}
& L_t u = f, \\
& u|_{t = 0} = u_0.
\end{align*}
\]

**Proof.** A solution \( u \) can be written

\[
u(t, x) = K(t, \tau) u_0(x) + \int_{\tau}^t K(t, \sigma) f(\sigma, x) d\sigma
\]

which is in \( \gamma_{2t-1}(R^n) \) with respect to \( x \), from Proposition 6.8 and 6.9.

For \( |t - \tau| \leq \delta \) and \( |\tau - \sigma| \leq \delta \), we can write

\[
K(t, \tau, \gamma, \tau, \gamma) = K(t, \tau, \gamma, \gamma) K(\tau, \gamma, \gamma),
\]

\[
= \sum_{p=1}^{l} \sum_{q=1}^{l} K^{(p)}(t, \tau, \gamma, \gamma) K^{(q)}(\tau, \gamma, \gamma)
\]

\[
+ K(t, \tau, \gamma, \gamma) K^{(q)}(\tau, \gamma, \gamma) + K^{(q)}(t, \tau, \gamma, \gamma) K(t, \gamma, \gamma)
\]

here \( K^{(p)}(t, \tau, \gamma, \gamma) = W^{(p)}(t, \tau), p = 1, \ldots, l \) and \( K^{(q)}(t, \tau, \gamma) = W^{(q)}(t, \tau) + \int_{\tau}^t W(t, \sigma) F(\sigma, \gamma) d\sigma \).

Since \( K^{(q)}(t, \tau, \gamma, \gamma) \) is in \( \gamma_{2t-1} \) with respect to \( x \) and \( y \), it follows from Proposition 6.8 and 6.9 that the wave front sets in \( \gamma_{2t-1} \) of \( K^{(q)}(t, \tau, \gamma, \gamma) K(\tau, \gamma, \gamma) \) and \( K(t, \tau, \gamma, \gamma) K^{(q)}(t, \tau, \gamma, \gamma) \) are empty. Hence we have

\[
K(t, \tau, \gamma, \gamma) \equiv \sum_{p=1}^{l} \sum_{q=1}^{l} K^{(p)}(t, \tau, \gamma, \gamma) K^{(q)}(\tau, \gamma, \gamma), \quad (\text{mod } \gamma_{2t-1})
\]
for $|t-\tau| \leq \delta$ and $|\tau-\sigma| \leq \delta$.

**Theorem 7.8.** For $|t-\tau| \leq \delta$ and $|\tau-\sigma| \leq \delta$, we have

\[
\hat{K}_1^{(\nu)}(t, x, \sigma, y) = \sum_{p=q} K^{(p)}(t, x, \tau, \cdots) K^{(p)}(\tau, \cdots, \sigma, y) \equiv 0 \pmod{\gamma_{2\tau-1}}.
\]

**Proof.** Since $L_{t, 2} K^{(p)}(t, x, \tau, \sigma, y) \equiv 0 \pmod{\gamma_{2\tau-1}}$, we have

\[
L_{t, 2} \hat{K}_1^{(\nu)}(t, x, \sigma, y) \equiv 0 \pmod{\gamma_{2\tau-1}}
\]

for $|t-\tau| \leq \delta$. Since $|\tau-\sigma| \leq \delta$, we can put $t=\sigma$. By virtue of Proposition 7.7 it suffices to prove

\[
\hat{K}_1^{(\nu)}(\sigma, x, \sigma, y) \equiv 0 \pmod{\gamma_{2\tau-1}}.
\]

Then we have from (7.8)

\[
\hat{K}_1^{(\nu)}(\sigma, x, \sigma, y) \equiv \delta(x-y) - \sum_{p=1}^{l} K^{(p)}(\sigma, x, \tau, \cdots) K^{(p)}(\tau, \cdots, \sigma, y) \pmod{\gamma_{2\tau-1}}
\]

Hence it follows from Corollary 7.2 that

\[
WF_{\gamma_{2\tau-1}}(\hat{K}_1^{(\nu)}(\sigma, \cdot, \sigma, y)) \subset [(y, \xi); \xi \in R^n \setminus \{0\}].
\]

On the other hand it follows from Theorem 7.1, that the wave front set in $\gamma_{2\tau-1}$ of $K^{(p)}(\sigma, x, \sigma, \cdots) K^{(p)}(\tau, \cdots, \sigma, y)$ is contained in $l^{(p)}(\sigma, \tau), l^{(p)}(\tau, \sigma, y)$. Hence we have

\[
WF_{\gamma_{2\tau-1}}(\hat{K}_1^{(\nu)}(\sigma, \cdot, \sigma, y)) \subset \bigcup_{p=q} l^{(p)}(\sigma, \tau), l^{(p)}(\tau, \sigma, y).
\]

From Lemma 7.6 it follows that

\[
\bigcup_{p=q} l^{(p)}(\sigma, \tau), l^{(p)}(\tau, \sigma, y) \cap [(y, R^n \setminus \{0\}) = \emptyset.
\]

Hence we obtain (7.10).

**Corollary 7.9.** For $0 < |\tau-\sigma| \leq \delta$, we have, $p=1, \cdots, l$.

\[
K^{(p)}(t, x, \tau, \cdot) K^{(p)}(\tau, \cdots, \sigma, y) \equiv K^{(p)}(t, x, \sigma, y) \pmod{\gamma_{2\tau-1}}.
\]

**Proof.** It suffices to prove (7.11) for $t=\tau$. Then from (7.9) and (7.10) we obtain

\[
\sum_{q=1}^{l} (K^{(q)}(\tau, x, \tau, \cdot) K^{(q)}(\tau, \cdots, \sigma, y) \equiv \sum_{q=1}^{l} K^{(q)}(\tau, x, \sigma, y).
\]

Hence

\[
K^{(p)}(\tau, x, \tau, \cdot) K^{(p)}(\tau, \cdots, \sigma, y) = K^{(p)}(\tau, x, \sigma, y) + \sum_{q=p}^{l} K^{(q)}(\tau, x, \tau, \cdot) K^{(q)}(\tau, \cdots, \sigma, y) - K^{(q)}(\tau, x, \sigma, y).
\]
It follows from (7.6) that

\[ A^{(p)}(\tau, \sigma; y) = \phi, \]

which implies (7.11).

We put, \( \sigma \leq t \leq \tau, \)

\[ S^{(p)}(t, x, \sigma, y) = K^{(p)}(t, x, \tau, \cdot)K^{(p)}(\tau, \cdot, \sigma, y) - K^{(p)}(t, x, \sigma, y) \]

\[ = \int \int \int \left( \exp i \langle \mathbb{Z}^{(p)}(t, x, \tau, \xi) - z, \xi \rangle + i \langle \mathbb{Z}^{(p)}(\tau, z, \sigma, \eta) - y, \eta \rangle \right) \]

\[ \times w^{(p)}(t, x, \tau, \xi)w^{(p)}(\tau, z, \sigma, \eta)d\xi d\eta d\eta \]

\[ - \int \left( \exp i \langle \mathbb{Z}^{(p)}(t, x, \sigma, \xi) - y, \xi \rangle \right)w^{(p)}(t, x, \sigma, \xi)d\xi, \]

**Proposition 7.10.** Let \( u \) be in \( S'(\mathbb{R}^n) \). Then \( WF_{2\varepsilon-1}(S^{(p)}(t, \sigma)u) = \phi, p = 1, \ldots, l. \)

**Proof.** We put

\[ I_N(y, \zeta) = \left( \exp -i \langle x, \zeta \rangle \right) \chi_N(x)S^{(p)}(t, x, \sigma, y)dx, \]

which satisfies

\[ |D_y^m I_N(y, \zeta)| \leq C_m |\zeta|^{-N} A^N N! \varepsilon^{-l-1} |1 + |y||^{-m}, |x|| \leq m, \]

for any positive integer \( m \). For, it is true for \( |y| \leq r, r \) is a positive constant. If \( r \) is suitably large, for \( |y| \leq r \) and for \( x \in \text{supp} \chi_N \), we have

\[ d_{\xi \tau}(\langle \mathbb{Z}^{(p)}(t, x, \tau, \xi) - z, \xi \rangle + \langle \mathbb{Z}^{(p)}(\tau, z, \sigma, \eta) - y, \eta \rangle - \langle x, \zeta \rangle) \neq 0, \]

\[ d_{\xi \tau}(\langle \mathbb{Z}^{(p)}(t, x, \sigma, \xi) - y, \xi \rangle - \langle x, \zeta \rangle) \neq 0, \]

where \( \zeta = \zeta/|\zeta| \). So we obtain (7.12) by part of integration. Hence

\[ |\langle I_N(\cdot, \cdot), u \rangle| \leq C_m \sup_y (1 + |y|)^m \sum_{|\alpha| \leq m} |D_y^m I_N(y, \zeta)| \]

\[ \leq C_m |\zeta|^{-N} A^N N! \varepsilon^{-l-1}. \]

when \( m \) is the order of the distribution \( u \).

Now we turn to prove (7.3) by induction with respect to \( j \). It is true for \( j = 1 \) from Theorem 7.8. Assume that (7.3) is valid for \( j-1 \). By virtue of Proposition 7.7 it suffices to prove that (7.3) is valid for \( t = t_j \). For, \( L_{t_j}K_j^{(0)}(t, x, t_j, y) = 0 \), for \( t_j \leq t \leq t_{j-1} \). We have from (7.1),

\[ K_j^{(0)}(t_j, t_0) = K(t_j, t_0) - \sum_{p=1}^{l} K_j^{(p)}(t_j, t_0) \]
\[ \equiv \sum_{p=1}^{i} K_{j}^{(p)}(t_{j}, t_{0}) - K_{j}^{(p)}(t_{j}, t_{0}) \pmod{\gamma_{2-1}} \]
\[ = -\sum_{p=1}^{i} (K^{(p)}(t_{j}, t_{j}) K^{(p)}(t_{j}, t_{j-1}) - K^{(p)}(t_{j}, t_{j-1}) K^{(p)}(t_{j-1}, t_{0})) \]
\[ = -\sum_{p=1}^{i} S^{(p)}(t_{j}, t_{j-1}) K_{j}^{(p)}(t_{j-1}, t_{0}) \]

of which wave front set in \( \gamma_{2-1} \) is empty from Proposition 7.10.

References

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