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A CONDITION FOR ALGEBRAS ASSOCIATED WITH A CYCLIC QUIVER TO BE SYMMETRIC

By

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Abstract. Let $K$ be a field, $f(x)$ a monic polynomial in $K[x]$ and $K\Gamma$ the path algebra of a cyclic quiver $\Gamma$ with $s$ vertices and $s$ arrows. In this paper, we give a necessary and sufficient condition for the algebra $K\Gamma/(f(X))$ to be a symmetric algebra, where $X$ is the sum of all arrows in $K\Gamma$.

1. Introduction

Let $K$ be a field and $\Gamma$ the cyclic quiver with $\{e_1, \ldots, e_s\}$ as the set of vertices and $\{a_1, \ldots, a_s\}$ as the set of arrows ($s \geq 2$) such that the start point and the end point of $a_t$ are $e_t$ and $e_{t+1}$, respectively. Let $K\Gamma$ be the path algebra of $\Gamma$. We denote the sum of all arrows by $X : X = a_1 + \cdots + a_s$. It is known by Erdmann and Holm [EH] that $K\Gamma/(X^p)$ is a symmetric algebra if and only if $p \equiv 1 \pmod{s}$. In this paper, we consider the $K$-algebra $A := K\Gamma/(f(X))$ where $f(x)$ is a monic polynomial over $K$. Our purpose is to give a necessary and sufficient condition for $A$ to be a symmetric algebra.

We describe the brief way to get the main theorem. First we will show that the equation $(f(X)) = (X^c h(X))$ holds where $h(x)$ is a monic polynomial in $K[x^s]$ and $c$ is an integer such that $0 \leq c \leq s - 1$. Second we construct a left $A$-isomorphism $\text{Hom}_K(A, K) \to A$ and also a right one (Propositions 2.3, 2.5). So we see that $A$ is a Frobenius algebra. If $c = 0$ and the constant term of $h(x)$ is nonzero, then we have a certain left $A$-isomorphism $A \to A$ and also a right one (Lemma 3.3). By the above propositions and lemma, we have an isomorphism $\text{Hom}_K(A, K) \to A$ of $A$-bimodules. Also if $c = 1$, then $A$ is a symmetric algebra; if $2 \leq c \leq s - 1$, then $A$ is a nonsymmetric algebra (Proposition 3.5). Summarizing these statements we get the following main result; $A$ is a

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symmetric algebra if and only if either \( c = 0 \) and the constant term of \( h(x) \) is nonzero or \( c = 1 \) holds (Theorem 3.6). Moreover, by means of the decomposition of algebras, we can compute the Hochschild cohomology ring of \( A \) in principle (Remark 3.8).

2. \( A \) is a Frobenius Algebra

Let \( s \) be a positive integer (\( s \geq 2 \)). By \( \Gamma \) we denote the cyclic quiver with \( \{e_1, \ldots, e_s\} \) as the set of vertices and \( \{a_1, \ldots, a_s\} \) as the set of arrows such that the start point and the end point of \( a_t \) are \( e_t \) and \( e_{t+1} \), respectively. Let \( K \) be a field and \( \mathcal{K} \Gamma \) the path algebra of \( \Gamma \). Here we regard the index \( t \) of \( e_t \) modulo \( s \). Hence \( a_t = e_{t+1}a_te_t \) holds for \( 1 \leq t \leq s \) in \( \mathcal{K} \Gamma \). We denote the sum of all arrows by \( X : X = a_1 + \cdots + a_s \). Then \( X^j \) is a sum of all paths of length \( j \) for \( j \geq 0 \).

Let \( f(x) \) be a monic polynomial of degree \( m \) (\( m \geq 1 \)) over \( K : f(x) = x_0 + x_1x + \cdots + x_{m-1}x^{m-1} + x^m \). We consider the \( K \)-algebra \( A = \mathcal{K} \Gamma / (f(X)) \).

For each \( i \) (\( 0 \leq i \leq s - 1 \)), we set

\[
 f_i(x) = x_i x^j + x_{i+j} x^{j+i} + x_{2i+j} x^{2j+i} + \cdots ,
\]

which is the sum of the all terms of \( f(x) \) whose degree is congruent to \( i \) modulo \( s \). Then \( f_i(x) \neq 0 \), and we set \( g_1(x) = 0 \) if \( f_i(x) = 0 \). Then \( g(x) := \gcd(g_0(x), g_1(x), \ldots, g_{s-1}(x)) \) is in \( K[x^s] \) since \( g_i(x) \in K[x^s] \). If we set \( d = \min \{n_i, s+i\} \) (\( 0 \leq i \leq s-1 \), \( f_i(x) \neq 0 \)), then there exist an integer \( c \) (\( 0 \leq c \leq s - 1 \)) and a monic polynomial \( h(x) \in K[x^s] \) such that \( \gcd(f_0(x), f_1(x), \ldots, f_{s-1}(x)) = x^d g(x) = x^c h(x) \). Note that \( c \) and \( h(x) \) are uniquely determined by \( f(x) \). Since \( e_{t+i} f(X) e_t = e_{t+i} f_i(X) e_t \) (\( 1 \leq t \leq s, 0 \leq i \leq s - 1 \)), we have the following equation of ideals in \( \mathcal{K} \Gamma \)

\[
 (f(X)) = (f_0(X)) + (f_1(X)) + \cdots + (f_{s-1}(X)) = (X^c h(X)).
\]

Thus we have the following lemma.

Lemma 2.1. For the algebra \( A \), there exist an integer \( c \) (\( 0 \leq c \leq s - 1 \)) and a monic polynomial \( h(x) \in K[x^s] \) such that

\[
 A = \mathcal{K} \Gamma / (X^c h(X)).
\]

Example 2.2. Let \( K \) be the field of rationals \( \mathbb{Q} \).

(i) Case \( s = 2 \). If \( f(x) = x - 2x^2 + x^3 \), then

\[
 (f(X)) = (f_0(X)) + (f_1(X)) + (f_2(X)) = (X^2 h(X)).
\]
Algebras associated with cyclic quiver

\[ f_0(x) = x^2g_0(x) = x^2 \cdot (-2), \]

\[ f_1(x) = xg_1(x) = x(1 + x^2). \]

Since \( \text{gcd}(f_0(x), f_1(x)) = x \), we have

\[ \mathbb{Q}\Gamma/(f(X)) = \mathbb{Q}\Gamma/(X^2). \]

(ii) Case \( s = 3 \). If \( f(x) = x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x + x^{10}, \) then

\[ f_0(x) = x^3g_0(x) = x^3(1 + x^3 + x^6), \]

\[ f_1(x) = x^4g_1(x) = x^4(1 + x^3 + x^6), \]

\[ f_2(x) = x^2g_2(x) = x^2(1 + x^3 + x^6). \]

Therefore \( \text{gcd}(f_0(x), f_1(x), f_2(x)) = x^2(1 + x^3 + x^6), \) so we have

\[ \mathbb{Q}\Gamma/(f(X)) = \mathbb{Q}\Gamma/(X^2(1 + X^3 + X^6)). \]

(iii) Case \( s = 4 \). If \( f(x) = x^5 - x^6 + x^7 + 2x^9 + 2x^{11} + 2x^{13} + x^{14} + 2x^{15} + x^{17} + 2x^{18} + 2x^{19} + x^{22} + x^{23} + x^{27}, \) then we write \( f(x) \) as follows:

\[
\begin{align*}
  f(x) &= x^5 + 2x^9 + 2x^{13} + x^{17} + (-x^6 + x^{14} + 2x^{18} + x^{22}) \\
  &= f_1(x) \\
  &+ x^7 + 2x^{11} + 2x^{15} + 2x^{19} + x^{23} + x^{27} \\
  &= f_3(x) \\
  &+ f_2(x) \\
  &= f_3(x).
\end{align*}
\]

Each of the above polynomials \( f_i(x) \) factors as follows:

\[ f_1(x) = x^5g_1(x) = x^5(1 + x^4)(1 + x^4 + x), \]

\[ f_2(x) = x^6g_2(x) = x^6(-1 + x^4 + x^8)(1 + x^4 + x^8), \]

\[ f_3(x) = x^7g_3(x) = x^7(1 + x^4 + x^{12})(1 + x^4 + x^8). \]

Therefore \( \text{gcd}(f_1(x), f_2(x), f_3(x)) = x^5(1 + x^4 + x^8), \) so we have

\[ \mathbb{Q}\Gamma/(f(X)) = \mathbb{Q}\Gamma/(X^5(1 + X^4 + X^8)) \]

\[ = \mathbb{Q}\Gamma/(X(X^4 + X^8 + X^{12})). \]

Using the above notations, we set \( h(x) = k_0 + k_1x + \cdots + k_{n-1}x^{(n-1)} + x^n \in K[x^n]. \) We will show that the \( K \)-algebra \( A = K\Gamma/(X^n h(X)) \) is a Frobenius algebra. In the rest of this paper, we use a representative elements instead of
their residue classes. We take the set \( \{X^jei \mid 1 \leq i \leq s, 0 \leq j \leq ns + c - 1 \} \) as a \( K \)-basis of \( A \) and also the dual basis \( \{(X^jei)^* \in \text{Hom}_K(A, K) \mid 1 \leq i \leq s, 0 \leq j \leq ns + c - 1 \} \) (cf. [FS]). Then we obtain the following proposition. On that occasion we set \( k_n = 1 \) in the following.

**Proposition 2.3.** We have a left \( A \)-isomorphism \( \varphi : \text{Hom}_K(A, K) \rightarrow A \) defined by

\[
\varphi((X^jei)^*) = \sum_{\ell = m+1}^{n} k_{\ell}e_{\ell}X^{\ell/s+c-j-1} \quad \text{for} \ 1 \leq i \leq s, \ 0 \leq j \leq ns + c - 1,
\]

where \( m \) is the integer \((-1 \leq m \leq n - 1)\) such that \( j = ms + c + r \) \((0 \leq r \leq s - 1)\). So \( A \) is a Frobenius algebra.

We prepare the following lemma for the proof of the proposition.

**Lemma 2.4.** Let \( i, j, t, t', u \) be integers with \( 1 \leq i, u \leq s, 0 \leq j \leq ns + c - 1, 1 \leq t \leq n - 1 \) and \( 0 \leq t' \leq n - 1 \). Then for \( (X^jei)^* \in \text{Hom}_K(A, K) \), we have

\[
X(X^jei)^* = \begin{cases} 
-k_0(X^{ns-1}e_{i+1})^* & \text{if } j = 0, c = 0, \\
0 & \text{if } j = 0, c \neq 0,
\end{cases}
\]

\[
\begin{cases} 
(X^{ts-1}e_{i+1})^* - k_t(X^{ns-1}e_{i+1})^* & \text{if } j = ts, c = 0, \\
(X^{t's+c-1}e_{i+1})^* - k_t(X^{ns+c-1}e_{i+1})^* & \text{if } j = t's + c, c \neq 0, \\
(X^jei)^* & \text{otherwise},
\end{cases}
\]

\[
e_u(X^jei)^* = \begin{cases} 
(X^jei)^* & \text{if } u = i, \\
0 & \text{if } u \neq i.
\end{cases}
\]

**Proof.** Case \( c = 0 \); If \( j = 0 \), then for \( 0 \leq p \leq ns - 1 \) and \( 1 \leq q \leq s \),

\[
(\dagger) \quad (X(e_i)^*)(X^pe_q) = (e_i)^*(X^pe_qX) = (e_i)^*(X^{p+1}e_{q-1}).
\]

Here in case of \( p + 1 = ns \) and \( q - 1 \equiv i \ (\mod s) \), since \( X^{ns} = -k_0 - k_1X^s - \cdots - k_{n-1}X^{(n-1)s} \) in \( A \), we have \( (e_i)^*(X^{ns}e_i) = (e_i)^*(-k_0 - \cdots - k_{n-1}X^{(n-1)s})e_i = -k_0 \). Therefore

\[
(\text{equation } (\dagger)) = \begin{cases} 
-k_0 & \text{if } p + 1 = ns \text{ and } q - 1 \equiv i \ (\mod s), \\
0 & \text{otherwise}.
\end{cases}
\]

On the other hand \(-k_0(X^{ns-1}e_{i+1})^*(X^pe_q) = -k_0 \) if \( p = ns - 1 \) and \( q \equiv i + 1 \ (\mod s) \), \( 0 \) otherwise. Thus we have \( X(e_i)^* = -k_0(X^{ns-1}e_{i+1})^* \). If \( j = ts \), then for
0 ≤ p ≤ ns − 1 and 1 ≤ q ≤ s, \( (X^{(X^q e_i)})^* (X^p e_q) = (X^{(X^p e_q)})^* (X^{p+1} e_{q-1}) = 1 \) if \( p + 1 = ts \) and \( q - 1 ≡ i \) (mod s), \(-k_i\) if \( p + 1 = ns \) and \( q - 1 ≡ i \) (mod s), 0 otherwise. Also \((X^{(n-1)e_{i+1}})^* - k_i (X^{(n-1)e_{i+1}})^*) (X^p e_q) = 1 \) if \( p = ts - 1 \) and \( q ≡ i + 1 \) (mod s), \(-k_i\) if \( p = ns - 1 \) and \( q ≡ i + 1 \) (mod s), 0 otherwise. Thus we have \( (X^{(X^p e_q)})^* = (X^{(X^{(n-1)e_{i+1}})})^* - k_i (X^{(n-1)e_{i+1}})^*) \).

Case \( c ≠ 0; \) If \( j = 0 \), then for \( 0 ≤ p ≤ ns + c - 1 \) and \( 1 ≤ q ≤ s \), \((X(e_i))^*(X^p e_q) = (X(e_i))^* (X^{p+1} e_q) = (X^{(X^p e_q)})^* \) if \( q = i \), 0 if \( q ≠ i \). Also we have \( (X(e_i))^* (X^p e_q) = (X_{X(e_i)})^* (X^{p+1} e_q) \) if \( q = i \), 0 if \( q ≠ i \). Hence we have \( e_u (X(e_i))^* = (X(e_i))^* \). If \( u ≠ i \), then for \( 0 ≤ p ≤ ns + c - 1 \) and \( 1 ≤ q ≤ s \), \((e_u (X(e_i))^*)(X^p e_q) = (X_{X(e_i)})^* (X^{p+1} e_q) \) if \( u = 0 \), 0 if \( q ≠ i \). Also we have \( (e_u (X(e_i))^*)(X^p e_q) = (X_{X(e_i)})^* (X^{p+1} e_q) \) if \( u ≠ 0 \), then we have \( e_u (X(e_i))^* = (X(e_i))^* \). Hence \( (e_u (X(e_i))^*)(X^p e_q) = 0 \) for \( 0 ≤ p ≤ ns + c - 1 \), \( 1 ≤ q ≤ s \). Therefore the proof of lemma is completed.

By this lemma, we will prove the Proposition 2.3.

**Proof of Lemma 2.3.** It is clear to see that \( \phi \) is an isomorphism of \( K \)-spaces. So it suffices to show that \( \phi \) is a homomorphism of left \( A \)-modules. Hence we prove that

\[
\phi(X(X^t e_i)^*) = X\phi((X^t e_i)^*), \quad (e_u (X(e_i))^*) = e_u \phi((X(e_i))^*),
\]

for \( 1 ≤ i, u ≤ s \) and \( 0 ≤ j ≤ ns + c - 1 \). First we will show that \( \phi(X(X^t e_i)^*) = X\phi(X(e_i))^* \). We consider the case \( c = 0 \). If \( j = 0 \), then we have \( X\phi((e_i)^*) = X\sum_{t=1}^{n} k_r e_t X^{(t-1)} = \sum_{t=1}^{n} k_r e_t X^{(t-1)} = -e_{t+1} \sum_{t=0}^{n-1} k_r X^{(t+s)} = -k_{t+1}, \) and \( \phi((X(e_i)^* = \phi(-k_0 (X^{(n-1)+1})^*) = -k_0 e_{t+1}. \) If \( j = ts \) (\( 1 ≤ t ≤ s - 1 \), then we have \( X\phi((X(e_i))^*) = X\sum_{t=1}^{n} k_r e_t X^{(t-1)s} = \sum_{t=1}^{n} k_r e_t X^{(t-1)s} = \sum_{t=1}^{n} k_r e_t X^{(t-1)s} = -k_{t+1}, \) and \( \phi((X(e_i))^* = \phi((X^{(n-1)+1})^* - k_1 (X^{(n-1)+1})^*) = \sum_{t=1}^{n} k_r e_t X^{(t-1)s} = 0. \) The remaining cases are clear. Therefore we have \( \phi(X(X^t e_i)^*) = X\phi((X^t e_i)^*). \)

Second we will show that \( \phi(e_u (X^t e_i)^*) = e_u \phi((X^t e_i)^*). \) If \( u = i \), then we have \( e_u \phi((X^t e_i)^*) = \sum_{t=1}^{n} k_r e_t X^{(t-1)s} = \phi((X^t e_i)^*) = \phi(e_u (X^t e_i)^*). \) If \( u ≠ i \), then
we have \(e_u \varphi((X^le_l)^*) = 0\) since \(e_u \neq e_l\). Also \(\varphi(e_u(X^le_l)^*) = \varphi(0) = 0\). Hence \(\varphi\) is an isomorphism of left \(A\)-modules. This completes the proof of the proposition.

Similarly, considering the operation of \(A\) onto \(\text{Hom}_K(A, K)\) from the right, we get the following proposition.

**Proposition 2.5.** We have a right \(A\)-isomorphism \(\psi : \text{Hom}_K(A, K) \to A\) defined by

\[
\psi((X^le_l)^*) = \sum_{\ell=m+1}^{n} k_{\ell}e_{i+\ell-c}X^{i+c-j-1} \text{ for } 1 \leq i \leq s, 0 \leq j \leq ns + c - 1,
\]

where \(m\) is the integer \((-1 \leq m \leq n - 1)\) such that \(j = ms + c + r \quad (0 \leq r \leq s - 1)\).

### 3. Main Theorem

In this section we give a necessary and sufficient condition for the algebra \(A = K\Gamma/(X^c h(X))\) to be a symmetric algebra, where \(c\) is the integer such that \(0 \leq c \leq s - 1\) and \(h(x) = k_0 + k_1x^3 + \cdots + x^{ms}\). We prepare some lemmas for the proof of the main theorem.

The following fact is described in [EH].

**Lemma 3.1.** \(K\Gamma/(X^p)\) (\(p \geq 1\)) is a symmetric algebra if and only if \(p \equiv 1 \pmod{s}\).

**Proof.** We denote \(K\Gamma/(X^p)\) by \(B\). We set \(p = ns + c\) (\(0 \leq c \leq s - 1\)) and \(h(x) = x^{ms}\). Then the above \(A\) coincides with \(B\). If \(p \equiv 1 \pmod{s}\), that is, \(c = 1\), then \(\varphi\) of Proposition 2.3 coincides with \(\psi\) of Proposition 2.5. Hence \(B\) is a symmetric algebra. Conversely we assume that \(B\) is a symmetric algebra. We will use an indirect proof by assuming that \(p \not\equiv 1 \pmod{s}\). Let \(\xi\) be an isomorphism of \(B\)-bimodules \(\text{Hom}_K(B, K) \to B\). Fix an \(i\) with \(1 \leq i \leq s\). Let \(\xi((e_i)^*) = \sum_{j=0}^{p-1} \sum_{\ell=1}^{s} k_{j,\ell}X^\ell e_\ell\) for \(k_{j,\ell} \in K\). Since \(\xi\) is an isomorphism of \(B\)-bimodules, the equation \(\xi((e_i)^*)e_u = \xi((e_i)^*e_u)\) holds for any \(1 \leq u \leq s\). The left hand side equals \(\sum_{j=0}^{p-1} \sum_{\ell=1}^{s} k_{j,\ell}X^\ell e_\ell e_u = \sum_{j=0}^{p-1} k_{j,u}X^\ell e_\ell\) and the right hand side equals \(\sum_{j=0}^{p-1} \sum_{\ell=1}^{s} k_{j,\ell}X^\ell e_\ell\) if \(i = u\), \(0\) if \(i \neq u\). This implies that \(k_{j,\ell} = 0\) for \(1 \leq \ell \leq s\) such that \(\ell \neq i\) and any \(0 \leq j \leq p - 1\). So we have \(\xi((e_i)^*) = \sum_{j=0}^{p-1} k_{j,i}X^\ell e_\ell\). Furthermore, the equation \(e_u \xi((e_i)^*) = \xi(e_u(e_i)^*)\) holds for \(1 \leq u \leq s\). The left hand side equals \(\sum_{j=0}^{p-1} k_{j,i}X^\ell e_u e_\ell = \sum_{j=0}^{p-1} k_{j,i}X^\ell e_\ell\) if
the similar way, we have the following lemma.

If \( p(x) \in K[x] \) and \( p(x) \) is not divided by \( x \), then we have the following decomposition of algebras for the algebra \( K \Gamma/(X'p(X)) \) \((r \geq 1)\):

\[
K \Gamma/(X'p(X)) \cong K \Gamma/(X') \oplus K \Gamma/(p(X)).
\]

**Proof.** Since \( x' \) and \( p(x) \) are relatively prime, we have \( X'u_1(X) + p(X)u_2(X) = 1 \) in \( K \Gamma \) for some \( u_1(x), u_2(x) \in K[x] \). Let \( z \in \langle X' \rangle \cap \langle p(X) \rangle \). If \( p(X) \in K[X'] = Z(K \Gamma) \), then there exist \( v_1, v_2 \in K \Gamma \) such that \( z = X'v_1 = p(X)v_2 \). So we have \( z = z(X'u_1(X) + p(X)u_2(X)) = v_2X'p(X)u_1(X) + X'p(X)v_1u_2(X) \in \langle X'p(X) \rangle \). Thus we have \( \langle X' \rangle \cap \langle p(X) \rangle \subset \langle X'p(X) \rangle \). The converse inclusion is clear. By Chinese remainder theorem, we have the decomposition

\[
K \Gamma/(X'p(X)) = K \Gamma/(\langle X' \rangle \cap \langle p(X) \rangle) \cong K \Gamma/(X') \oplus K \Gamma/(p(X)).
\]

**Lemma 3.3.** Let \( c = 0 \). If \( k_0 \neq 0 \), then we have a left \( A \)-isomorphism \( \phi' : A \rightarrow A \) defined by \( \phi'(e_1X^j) = e_1X^{j+1} \) and a right \( A \)-isomorphism \( \psi' : A \rightarrow A \) defined by \( \psi'(e_1X^j) = e_{i+1}X^{j+1} \) for \( 1 \leq i \leq s, 0 \leq j \leq ns - 1 \).

**Proof.** Since \( k_0 \neq 0 \), each \( K \)-linear maps is an isomorphism of \( K \)-spaces. It is easy to show that these maps are homomorphisms of \( A \)-modules.

**Proposition 3.4.** Let \( c = 0 \). Then \( A \) is a symmetric algebra if and only if \( k_0 \neq 0 \).
Proof. If $k_0 \neq 0$, then by Propositions 2.3, 2.5 and Lemma 3.3, we have the left $A$-isomorphism $\varphi' \circ \psi : \text{Hom}_K(A, K) \to A((X^i e_i)^*) \ni \sum_{\ell=m+1}^n k_{i\ell} X^{\ell-i}$, and also we have the right $A$-isomorphism $\psi' \circ \psi : \text{Hom}_K(A, K) \to A((X^i e_i)^*) \ni \sum_{\ell=m+1}^n k_{i\ell} X^{\ell-i}$. Thus $\varphi' \circ \psi$ coincides with $\psi' \circ \psi$, so this is the isomorphism of $A$-bimodules. This means that $A$ is a symmetric algebra. Conversely we assume that $k_0 = 0$. Then there exists an integer $t$ ($1 \leq t \leq n$) such that $h(x) = x^t h_0(x)$ where the constant term of $h_0(x) \in K[x^t]$ is nonzero. By Lemma 3.2, we have the following decomposition of $A$:

$$A = K\Gamma/(X^{ts+i}) \oplus K\Gamma/(h_0(X)).$$

For the decomposition, $K\Gamma/(X^{ts+i})$ is a nonsymmetric algebra by Lemma 3.1. Hence $A$ is a nonsymmetric algebra too ([EN, Proposition 1]). This completes the proof of the lemma.

Proposition 3.5. If $c = 1$, then $A$ is a symmetric algebra, and if $2 \leq c \leq s - 1$, then $A$ is a nonsymmetric algebra.

Proof. For the algebra $A$, there exists the integer $t$ ($0 \leq t \leq n$) such that $(X^t h(X)) = (X^{ts+i} h_0(X))$ where the constant term of $h_0(x) \in K[x^t]$ is nonzero. Then, by Lemma 3.2, we have the following decomposition:

$$A = K\Gamma/(X^{ts+i} h_0(X)) \cong K\Gamma/(X^{ts+c}) \oplus K\Gamma/(h_0(X)).$$

By Proposition 3.4, $K\Gamma/(h_0(X))$ is a symmetric algebra. By Lemma 3.1, if $c = 1$, then $K\Gamma/(X^{ts+i})$ is a symmetric algebra, and if $2 \leq c \leq s - 1$, then $K\Gamma/(X^{ts+c})$ is a nonsymmetric algebra.

We summarize the above results as follows.

Theorem 3.6. $A$ is a symmetric algebra if and only if either $c = 0$ and $k_0 \neq 0$ hold or $c = 1$ holds.

Example 3.7. In Example 2.2, the algebras of the cases (i), (iii) are symmetric algebras, but one of the case (ii) is a nonsymmetric algebra.

Remark 3.8. We saw that there is a decomposition $A = K\Gamma/(X^{ts+c} h_0(X)) \cong K\Gamma/(X^{ts+c}) \oplus K\Gamma/(h_0(X))$ where the constant term of $h_0(x) \in K[x^t]$ is nonzero and $0 \leq c \leq s - 1$. For the decomposition of $A$, the Hochschild cohomology ring
of the first term is given by [EH], and also one of the second term is given by [FS]. Therefore the Hochschild cohomology ring of \( A \) is obtained by these facts.

For example, we denote the \( \mathbb{Q}\Gamma / \langle X^2(1 + X^3 + X^6) \rangle \) (\( \simeq \mathbb{Q}\Gamma / (X^2) \oplus \mathbb{Q}\Gamma / (1 + X^3 + X^6) \)) in Example 2.2 (ii) by \( C \). We will compute the even Hochschild cohomology ring \( \text{HH}^{ev}(C) = \bigoplus_{i \geq 0} \text{HH}^{2i}(C) \). By [EH, Section 4.8], the even Hochschild cohomology ring \( \text{HH}^{ev}(\mathbb{Q}\Gamma / (X^2)) \) is isomorphic to \( \mathbb{Q}[y_2, y_6]/(y_2^2, y_2y_6) \) where \( \deg y_2 = 2 \) and \( \deg y_6 = 6 \). Also, by [FS, Propositions 3.2, 3.7], the even Hochschild cohomology ring \( \text{HH}^{ev}(\mathbb{Q}\Gamma / (1 + X^3 + X^6)) \) is isomorphic to \( \mathbb{Q}[z_0]/(1 + z_0 + z_0^2) \) where \( \deg z_0 = 0 \). Thus we have

\[
\text{HH}^{ev}(C) \simeq \mathbb{Q}[y_2, y_6]/(y_2^2, y_2y_6) \oplus \mathbb{Q}[z_0]/(1 + z_0 + z_0^2).
\]

References


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