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Axiomatic Differential Geometry II-3
-Its Developments-
Chapter 3: The General Jacobi Identity

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Abstract
As the fourth paper of our series of papers concerned with axiomatic differential geometry, this paper is devoted to the general Jacobi identity supporting the Jacobi identity of vector fields. The general Jacobi identity can be regarded as one of the few fundamental results belonging properly to smootheology.

1 Introduction

It is well known in traditional differential geometry that the totality of vector fields on a smooth manifold forms a Lie algebra. The proof of this fact is tremendously easy, because we can identity vector fields with derivations within the particular category that orthodox differential geometers have indulged in. An axiomatic treatment of differential geometry emancipates differential geometers from this comfortable adherence to their favorite category of smooth manifolds and forces them to confront the infinitesimal structure per se barehanded.

The Jacobi identity occupies the central position in the structure of a Lie algebra, and we stumbled upon the general Jacobi identity supporting the Jacobi identity of vector fields from behind the very vale of infinitesimal structures within the framework of synthetic differential geometry in the previous century, for which the reader is referred to [2], [3] and [4]. This paper is devoted to the general Jacobi identity within our axiomatics of differential geometry, which will play a predominant role in a subsequent paper dealing with the Frölicher-Nijenhuis calculus.

Our axiomatic differential geometry is an attempt to grasp the infinitesimal structure without fringes or frills. It seems that the term "smootheology" or "diffeology" is gaining momentum for the study of such an infinitesimal structure. We think that the general Jacobi identity is one of the few fundamental
results indigenous to smootheology. This infinitesimal structure lies at the very
core of not only differential geometry but also many pure or applied branches of
mathematics. We assume that the reader is familiar with our axiomatic frame-
work of differential geometry presented in [7] and [9]. Thus we are working
within a DG-category
\((\mathcal{K}, \mathbb{R}, T, \alpha)\)
in the sense of [9]. We always assume that \(M\) is a microlinear and Weil-
exponentiable object in the category \(\mathcal{K}\).

The general Jacobi identity will be dealt with in Section 4, which will be
preceded by the more elementary treatment of the primordial general Jacobi
identity in Section 3. The final section is devoted to the derivation of the Jacobi
identity of vector fields from the general Jacobi identity, in which the reader is
assumed to be familiar with [8].

2 Simplicial Sets

We need to fix notation and terminology for simplicial objects, which form an
important subclass of infinitesimal objects. Simplicial objects are infinitesimal
objects of the form

\[
D^n(p) = \{(d_1, ..., d_n) \in D^n \mid d_{i_1} ... d_{i_k} = 0 \ (\forall (i_1, ..., i_k) \in p)\}
\]

where \(p\) is a finite set of finite sequences \((i_1, ..., i_k)\) of natural numbers between
1 and \(n\), including the endpoints, with \(i_1 < ... < i_k\). If \(p\) is empty, \(D^n\{p\}\) is
\(D^n\) itself. If \(p\) consists of all the binary sequences, then \(D^n(p)\) represents \(D(n)\)
in the standard terminology of SDG. Given two simplicial objects \(D^m(p)\) and
\(D^n(q)\), we define a simplicial object \(D^m(p) \oplus D^n(q)\) to be

\[
D^{m+n}(p \oplus q)
\]

where

\[
p \oplus q = p \cup \{(j_1 + m, ..., j_k + m) \mid (j_1, ..., j_k) \in q\}
\]
\[
\cup \{(i, j + m) \mid 1 \leq i \leq m, \ 1 \leq j \leq n\}
\]

Since the operation \(\oplus\) is associative, we can combine any finite number of
simplicial objects by \(\oplus\) without bothering about how to insert parentheses. Given
morphisms of simplicial objects \(\Phi_i : D^{m_i}(p_i) \to D^m(p)\) \((1 \leq i \leq n)\), there exists
a unique morphism of simplicial objects \(\Phi : D^{m_1}(p_1) \oplus ... \oplus D^{m_n}(p_n) \to D^m(p)\)
whose restriction to \(D^{m_i}(p_i)\) coincides with \(\Phi_i\) for each \(i\). We denote this \(\Phi\) by
\(\Phi_1 \oplus ... \oplus \Phi_n\). We write \(D(n)\) for \(\{(d,...,d) \in D^n \mid d_i d_j = 0 \text{ for any } i \neq j\}\).
3 The Preliminary Identity

The principal objective in this paper is to give the general Jacobi identity and its proof. Our harder treatment of the general Jacobi identity in the coming section is preceded by a simpler treatment of the primordial general Jacobi identity in this section, because the latter is easy to grasp intuitively so that it prepares the reader for the coming general Jacobi identity.

Proposition 1 The diagram

\[
\begin{array}{ccc}
M \otimes W_{D^3 \{ (1,3), (2,3) \}} & \xrightarrow{id_M \otimes W_\varphi} & M \otimes W_{D^2} \\
\downarrow & & \downarrow \\
M \otimes W_{D^2} & \xrightarrow{id_M \otimes W_{D^2 \{ D(2) \}}} & M \otimes W_{D(2)} \\
\end{array}
\]

is a pullback diagram, where the assumptive mapping \( \varphi : D^2 \rightarrow D^3 \{ (1,3), (2,3) \} \) is

\[(d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0) \in D^3 \{ (1,3), (2,3) \}\]

while the assumptive mapping \( \psi : D^2 \rightarrow D^3 \{ (1,3), (2,3) \} \) is

\[(d_1, d_2) \in D^2 \mapsto (d_1, d_2, d_1d_2) \in D^3 \{ (1,3), (2,3) \}\]

Proof. This follows from the microlinearity of \( M \) and the pullback diagram of Weil algebras

\[
\begin{array}{ccc}
W_{D^3 \{ (1,3), (2,3) \}} & \xrightarrow{W_\varphi} & W_{D^2} \\
\downarrow & & \downarrow \\
W_{D^2} & \xrightarrow{W_{D^2 \{ D(2) \}}} & W_{D(2)} \\
\end{array}
\]

Corollary 2 We have

\[M \otimes W_{D^3 \{ (1,3), (2,3) \}} = (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2})\]

Notation 3 We will write

\[\zeta^\prime : (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2}) \rightarrow M \otimes W_D\]

for the morphism

\[
\left( id_M \otimes W_{d \in D \rightarrow (0,0,d) \in D^3 \{ (1,3), (2,3) \}}, W_{D^2 \{ D(2) \}} \circ W_\varphi \right)
\]

: \( W_{D^2} \times W_{D(2)} W_{D^2} = W_{D^2} \times W_{D(2)} W_{D^2} \rightarrow M \otimes W_D \)
The following is the prototype for the general Jacobi identity.

**Theorem 4 (The Primordial General Jacobi Identity)** The three morphisms

\[
\begin{align*}
\zeta^{*2\to3} & : (M \otimes W_{D^2}) \times M \otimes W_{D^2} \times (M \otimes W_{D^2}) \\
& \quad \rightarrow (M \otimes W_{D^2}) \times (M \otimes W_{D^2}) \times (M \otimes W_{D^2}) \\
\end{align*}
\]

\[
\begin{align*}
\zeta^{*3\to1} & : (M \otimes W_{D^2}) \times M \otimes W_{D^2} \times (M \otimes W_{D^2}) \\
& \quad \rightarrow (M \otimes W_{D^2}) \times (M \otimes W_{D^2}) \times (M \otimes W_{D^2}) \\
\end{align*}
\]

\[
\begin{align*}
\zeta^{*1\to2} & : (M \otimes W_{D^2}) \times M \otimes W_{D^2} \times (M \otimes W_{D^2}) \\
& \quad \rightarrow (M \otimes W_{D^2}) \times (M \otimes W_{D^2}) \times (M \otimes W_{D^2}) \\
\end{align*}
\]

sum up only to vanish, where the numbers under \((M \otimes W_{D^2})\) are given simply so as for the reader to relate each occurrence of \((M \otimes W_{D^2})\) to another, and the unlabeled arrows are the canonical projections.

The proof of Theorem 4 is based completely upon the following theorem.

**Theorem 5** The diagram

\[
\begin{array}{ccc}
\text{id}_M \otimes W_{i_{D^2}} & & \text{id}_M \otimes W_{i_{D^2}} \\
\downarrow & \nearrow & \downarrow \\
\text{id}_M \otimes W_{i_{D^2}} & & \text{id}_M \otimes W_{i_{D^2}} \\
\end{array}
\]

\[
\begin{array}{ccc}
M \otimes W_{D^2} & & M \otimes W_{D^2} \\
\downarrow & \nearrow & \downarrow \\
M \otimes W_{D^2} & & M \otimes W_{D^2} \\
\end{array}
\]

\[
\begin{array}{ccc}
M \otimes W_{D^2} & & M \otimes W_{D^2} \\
\downarrow & \nearrow & \downarrow \\
M \otimes W_{D^2} & & M \otimes W_{D^2} \\
\end{array}
\]

is a limit diagram, where the assumptive object \(E\) is

\[D^4\{(1, 3), (2, 3), (1, 4), (2, 4), (3, 4)\}\]

and the assumptive mapping \(i_{D^2}^{D^2} : D^2 \rightarrow D^2\) is \((d_1, d_2) \in D(2) \mapsto (d_1, d_2) \in D^2\), while the three unnamed arrows \(M \otimes W_E \rightarrow M \otimes W_{D^2}\) are \(\text{id}_M \otimes W_{i_{D^2}}\) \((i = 1, 2, 3)\) counterclockwise from the top with the assumptive mappings \(l_i : D^2 \rightarrow E\) \((i = 1, 2, 3)\) being

\[
\begin{align*}
l_1 : (d_1, d_2) & \in D^2 \mapsto (d_1, d_2, 0, 0) \in E \\
l_2 : (d_1, d_2) & \in D^2 \mapsto (d_1, d_2, d_1d_2, 0) \in E \\
l_3 : (d_1, d_2) & \in D^2 \mapsto (d_1, d_2, 0, d_1d_2) \in E \\
\end{align*}
\]
Corollary 6 We have
\[(M \otimes W_{D^2}) \times M \otimes W_{D^2} \times M \otimes W_{D^2} = M \otimes W_E\]
This theorem follows directly from the following lemma.

Lemma 7 The following diagram is a limit diagram of Weil algebras:
\[
\begin{array}{ccc}
W_{i_{D^2}} & W_{D^2} & W_{i_{D^2}} \\
\downarrow \swarrow & \downarrow \searrow \swarrow & \downarrow \searrow \\
W_{D^2} & W_E & W_{D^2} \\
\uparrow \swarrow & \downarrow \searrow & \downarrow \swarrow \\
W_{i_{D^2}} & W_{D^2} & W_{i_{D^2}}
\end{array}
\]

Proof. Let \(\gamma_1, \gamma_2, \gamma_3 \in W_{D^2}\) and \(\gamma \in W_E\) so that they are the polynomials with coefficients in \(k\) of the following forms:
\[
\begin{align*}
\gamma_1(X_1, X_2) &= a + a_1X_1 + a_2X_2 + a_{12}X_1X_2 \\
\gamma_2(X_1, X_2) &= b + b_1X_1 + b_2X_2 + b_{12}X_1X_2 \\
\gamma_3(X_1, X_2) &= c + c_1X_1 + c_2X_2 + c_{12}X_1X_2 \\
\gamma(X_1, X_2, X_3, X_4) &= e + e_1X_1 + e_2X_2 + e_{12}X_1X_2 + e_3X_3 + e_4X_4
\end{align*}
\]
The condition that \(W_{i_{D^2}}(\gamma_1) = W_{i_{D^2}}(\gamma_2) = W_{i_{D^2}}(\gamma_3)\) is equivalent to the following three conditions as a whole:
\[
\begin{align*}
a &= b = c \\
a_1 &= b_1 = c_1 \\
a_2 &= b_2 = c_2
\end{align*}
\]
Therefore, in order that \(W_1(\gamma) = \gamma_1, W_2(\gamma) = \gamma_2\) and \(W_3(\gamma) = \gamma_3\) in this case, it is necessary and sufficient that the polynomial \(\gamma\) should be of the following form:
\[
\gamma(X_1, X_2, X_3, X_4) = a + a_1X_1 + a_2X_2 + a_{12}X_1X_2 + (b_{12} - a_{12})X_3 + (c_{12} - a_{12})X_4
\]
This completes the proof.

Theorem 8 The diagram

\[
\begin{array}{ccc}
\text{id}_M \otimes W_\psi & M \otimes W_C & \text{id}_M \otimes W_\psi \\
\downarrow \swarrow & \downarrow \searrow & \downarrow \swarrow \\
M \otimes W_{D^2} & M \otimes W_E & M \otimes W_{D^2} \\
\uparrow \swarrow & \downarrow \searrow & \downarrow \swarrow \\
\text{id}_M \otimes W_\psi & M \otimes W_C & \text{id}_M \otimes W_\psi
\end{array}
\]
is a limit diagram, where $C$ stands for

$$D^3\{(1,3),(2,3)\}$$

and the three unnamed morphisms go clockwise from the top as follows:

- $\mathrm{id}_M \otimes \psi : M \otimes \psi \to \psi$,
- $\phi : \phi \to \phi$,
- $\psi : \psi \to \psi$.

This theorem follows directly from the following lemma.

**Lemma 9** The diagram

\[
\begin{array}{ccc}
\psi & \rightarrow & \psi \\
\downarrow & & \downarrow \\
\psi & \rightarrow & \psi \\
\downarrow & & \downarrow \\
\psi & \rightarrow & \psi
\end{array}
\]

is a limit diagram, where the three unnamed morphisms go clockwise from the top as follows:

- $\psi : (d_1,d_2,d_3) \in D^3\{(1,3),(2,3)\} \to (d_1,d_2,d_3,0) \in E$,
- $\phi : (d_1,d_2,d_3) \in D^3\{(1,3),(2,3)\} \to (d_1,d_2,d_3,0) \in E$,
- $\psi : (d_1,d_2,d_3) \in D^3\{(1,3),(2,3)\} \to (d_1,d_2,d_3,0) \in E$.

**Proof.** By the same token as in Lemma 7.

**Proof.** (of the primordial Jacobi identity). The morphism

\[\zeta : (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2}) \to M \otimes W_D\]

is the composition of

- $\mathrm{id}_M \otimes \psi : M \otimes \psi \to \psi$,
- $\phi : \phi \to \phi$,
- $\psi : \psi \to \psi$.

and

\[\zeta : (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2}) \to M \otimes W_D\]

in succession, which is in turn equivalent to

$\mathrm{id}_M \otimes \psi : (d_1,d_2,d_3,0) \in E$.
The morphism
\[ \zeta^{3\to2} : (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2}) \times_{M \otimes W_{D(3)}} (M \otimes W_{D^2}) \to M \otimes W_{D} \]
is the composition of
\[
\text{id}_M \otimes W_{d(d_1,d_2,d_3) \in D^3(1,3),(2,3)} \to (d_1,d_2,d_1d_2d_3,d_3) \in E
\]
\[ : (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2}) \times_{M \otimes W_{D(3)}} (M \otimes W_{D^2}) \to M \otimes W_C \]
\[ = (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2})^{3} \]
and
\[ \zeta^{-} : (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2})^{3} \to M \otimes W_{D} \]
in succession, which is in turn equivalent to
\[ \text{id}_M \otimes W_{d \in D \to (0,0,-d) \in E} \]
The morphism
\[ \zeta^{1\to3} : (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2}) \times_{M \otimes W_{D(3)}} (M \otimes W_{D^2}) \to M \otimes W_{D} \]
is the composition of
\[
\text{id}_M \otimes W_{d(d_1,d_2,d_3) \in D^3(1,3),(2,3)} \to (d_1,d_2,0,d_1d_2d_3) \in E
\]
\[ : (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2}) \times_{M \otimes W_{D(3)}} (M \otimes W_{D^2}) \to M \otimes W_C \]
\[ = (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2})^{1} \]
and
\[ (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2})^{1} \to M \otimes W_{D} \]
in succession, which is in turn equivalent to
\[ \text{id}_M \otimes W_{d \in D \to (0,0,0,-d) \in E} \]
Therefore
\[ \zeta^{2\to1} + \zeta^{3\to2} + \zeta^{1\to3} \]
\[ : (M \otimes W_{D^2}) \times_{M \otimes W_{D(2)}} (M \otimes W_{D^2}) \times_{M \otimes W_{D(3)}} (M \otimes W_{D^2}) \to M \otimes W_{D} \]
is equivalent to
\[
(\text{id}_M \otimes W_{d \in D \to (d,d,d) \in D(3)}) \circ (\text{id}_M \otimes W_{(d_1,d_2,d_3) \in D(3) \to (0,0,0,d_1d_2d_3,d_3) \in E})
\]
\[ = \text{id}_M \otimes (W_{d \in D \to (d,d,d) \in D(3)} \circ W_{(d_1,d_2,d_3) \in D(3) \to (0,0,0,d_1d_2d_3,d_3) \in E})
\]
\[ = \text{id}_M \otimes W_{d \in D \to (0,0,0,0) \in E} \]
This completes the proof. ■
4 The Main Identity

**Proposition 10** The diagram

\[
\begin{array}{ccc}
\text{id}_M \otimes W_{\phi^3_1} & \rightarrow & M \otimes W_{D^3} \\
\downarrow & & \downarrow \\
M \otimes W_{D^3} & \rightarrow & M \otimes W_{D^3(2,3)}
\end{array}
\]

is a pullback diagram, where the assumptive mapping \( \phi^3_1 : D^3 \rightarrow D^4\{2, 4\}, (3, 4) \) is

\[(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0) \in D^4\{2, 4\}, (3, 4) \]

while the assumptive mapping \( \psi^3_1 : D^3 \rightarrow D^4\{2, 4\}, (3, 4) \) is

\[(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, d_2d_3) \in D^4\{2, 4\}, (3, 4) \]

**Proof.** This follows from the microlinearity of \( M \) and the pullback diagram of Weil algebras

\[
\begin{array}{ccc}
W_{\phi^3_1} & \rightarrow & W_{D^3} \\
\downarrow & & \downarrow \\
W_D^3 & \rightarrow & W_{D^3(2,3)} \\
\text{id}_M \otimes W_{\phi^3_1} & \rightarrow & M \otimes W_{D^3(2,3)}
\end{array}
\]

**Corollary 11** We have

\[(M \otimes W_{D^3}) \times_{M \otimes W_{D^3(2,3)}} (M \otimes W_{D^3}) = M \otimes W_{D^4\{2, 4\}, (3, 4)} \]

with the diagrams

\[
\begin{array}{ccc}
(M \otimes W_{D^3})_1 \times_{M \otimes W_{D^3(2,3)}} (M \otimes W_{D^3})_2 & \rightarrow & M \otimes W_{D^3} \\
\| & & \| \\
M \otimes W_{D^3(2,4), (3, 4)} & \rightarrow & M \otimes W_{D^3(2,3)} \\
\text{id}_M \otimes W_{\phi^3_1} & \rightarrow & M \otimes W_{D^3(2,3)}
\end{array}
\]

and

\[
\begin{array}{ccc}
(M \otimes W_{D^3})_1 \times_{M \otimes W_{D^3(2,3)}} (M \otimes W_{D^3})_2 & \rightarrow & M \otimes W_{D^3} \\
\| & & \| \\
M \otimes W_{D^3(2,4), (3, 4)} & \rightarrow & M \otimes W_{D^3(2,3)} \\
\text{id}_M \otimes W_{\psi^3_1} & \rightarrow & M \otimes W_{D^3(2,3)}
\end{array}
\]

being commutative, where the unnamed arrows are canonical projections.
Notation 12 We will write
\[ \zeta : (M \otimes W_{D^3}) \times_{M \otimes W_{D^3} \{2,3\}} (M \otimes W_{D^3}) \to M \otimes W_{D^2} \]
for the morphism
\[
\begin{align*}
\text{id}_M \otimes W_{(d_1,d_2) \in D^2} & \to (d_1,0,0,d_2) \in D^4 \{ (2,4), (3,4) \} \\
(M \otimes W_{D^3}) \times_{M \otimes W_{D^3} \{2,3\}} (M \otimes W_{D^3}) & = M \otimes W_{D^4 \{ (2,4), (3,4) \}} \\
& \to M \otimes W_{D^2} 
\end{align*}
\]

Proposition 13 The diagram
\[
\begin{array}{ccc}
M \otimes W_{D^4 \{ (1,4), (3,4) \}} & \xrightarrow{id_M \otimes W_{\varphi^2}} & M \otimes W_{D^3} \\
id_M \otimes W_{\psi^2} \downarrow & & \downarrow & & \downarrow id_M \otimes W_{\psi^3_{D^3 \{ (1,3) \}}} \\
M \otimes W_{D^3} & \xrightarrow{id_M \otimes W_{\psi^3_{D^3 \{ (1,3) \}}}} & M \otimes W_{D^3 \{ (1,3) \}} 
\end{array}
\]
is a pullback diagram, where the assumptive mapping \( \varphi^3_{D^3} : D^3 \to D^4 \{ (1,4), (3,4) \} \) is
\[(d_1,d_2,d_3) \in D^3 \mapsto (d_1,d_2,d_3,0) \in D^4 \{ (1,4), (3,4) \}\]
while the assumptive mapping \( \psi^3_{D^3} : D^3 \to D^4 \{ (1,4), (3,4) \} \) is
\[(d_1,d_2,d_3) \in D^3 \mapsto (d_1,d_2,d_3,d_1d_3) \in D^4 \{ (1,4), (3,4) \}\]

Proof. This follows from the microlinearity of \( M \) and the pullback diagram of Weil algebras

\[\begin{array}{ccc}
W_{\varphi^3_{D^3 \{ (1,4), (3,4) \}}} & \to & W_{D^3} \\
\downarrow & & \downarrow \\
W_{D^3} & \to & W_{D^3 \{ (1,3) \}} \\
W_{\psi^3_{D^3 \{ (1,3) \}}} \end{array}\]

Corollary 14 We have
\[(M \otimes W_{D^3}) \times_{M \otimes W_{D^3} \{ (1,3) \}} (M \otimes W_{D^3}) = M \otimes W_{D^4 \{ (1,4), (3,4) \}}\]
with the diagrams
\[\begin{array}{ccc}
(M \otimes W_{D^3}) \times_{M \otimes W_{D^3} \{ (1,3) \}} (M \otimes W_{D^3}) & \to & M \otimes W_{D^3} \\
\text{id}_M \otimes W_{\varphi^3} & \| & \| \\
M \otimes W_{D^4 \{ (1,4), (3,4) \}} & \to & M \otimes W_{D^3} 
\end{array}\]
and
\[(M \otimes W_{D^3}) \times_{M \otimes W_{D^3}(1,3)} (M \otimes W_{D^3}) \rightarrow M \otimes W_{D^3}
\]
\[
\begin{array}{ccc}
M \otimes W_{D^3(D^{(1,3)})} & \otimes W_{D^3(D^{(1,3)})} & \rightarrow \ M \otimes W_{D^3
}
\end{array}
\]
\[
\begin{array}{ccc}
\text{id}_M \otimes W_{\psi_3^3} & \rightarrow & M \otimes W_{D^3(D^{(1,3)})}
\end{array}
\]
being commutative, where unnamed arrows are canonical projections.

**Notation 15** We will write
\[
\zeta^2 : (M \otimes W_{D^3}) \times_{M \otimes W_{D^3}(1,3)} (M \otimes W_{D^3}) \rightarrow M \otimes W_{D^2}
\]
for the morphism
\[
\begin{array}{ccc}
id_M \otimes W_{\psi_3^3} & \rightarrow & M \otimes W_{D^3(D^{(1,3)})}
\end{array}
\]
\[
\begin{array}{ccc}
(M \otimes W_{D^3}) \times_{M \otimes W_{D^3}(1,3)} (M \otimes W_{D^3}) & \rightarrow & M \otimes W_{D^2}
\end{array}
\]

**Proposition 16** The diagram
\[
\begin{array}{ccc}
id_M \otimes W_{\psi_3^3} & \rightarrow & M \otimes W_{D^3(D^{(1,3)})}
\end{array}
\]
is a pullback diagram, where the assumptive mapping \(\psi_3^3 : D^3 \rightarrow D^4\{(1, 4), (2, 4)\}\)
is
\[
(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0) \in D^4\{(1, 4), (2, 4)\}\]
while the assumptive mapping \(\psi_3^3 : D^3 \rightarrow D^4\{(1, 4), (2, 4)\}\) is
\[
(d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, d_1d_2) \in D^4\{(1, 4), (2, 4)\}\]

**Proof.** This follows from the microlinearity of \(M\) and the pullback diagram of Weil algebras
Corollary 17 We have
\[(M \otimes W^3) \times_{M \otimes W^3(1,2)} (M \otimes W^3) = M \otimes W^4_{1,2}\]
with the diagrams
\[
\begin{array}{ccc}
(M \otimes W^3) \times_{M \otimes W^3(1,2)} (M \otimes W^3) & \rightarrow & M \otimes W^3 \\
\| & & \| \\
M \otimes W^4_{1,2} & \rightarrow & M \otimes W^3 \\
\text{id}_M \otimes W^{3}_{2} & & \\
\end{array}
\]
and
\[
\begin{array}{ccc}
(M \otimes W^3) \times_{M \otimes W^3(1,2)} (M \otimes W^3) & \rightarrow & M \otimes W^3 \\
\| & & \| \\
M \otimes W^4_{1,2} & \rightarrow & M \otimes W^3 \\
\text{id}_M \otimes W^{3}_{2} & & \\
\end{array}
\]
being commutative, where unnamed arrows are canonical projections.

Notation 18 We will write \(\zeta \cdot -\cdot^3\) for the morphism
\[
\begin{array}{c}
(M \otimes W^3) \times_{M \otimes W^3(1,2)} (M \otimes W^3) \\
\rightarrow \\
M \otimes W^4_{1,2} \\
\end{array}
\]
for the morphism
\[
\begin{array}{c}
\text{id}_M \otimes W(d_1, d_2) \in D^2 \rightarrow (0, 0, d_1, 0, d_2) \in D^4(1,4,3,4) \\
(M \otimes W^3) \times_{M \otimes W^3(1,2)} (M \otimes W^3) \\
= M \otimes W^4_{1,2} \\
\rightarrow M \otimes W^2 \\
\end{array}
\]
Notation 19 We will write \(i^{14}_{14}, i^{24}_{24}\) and \(i^{34}_{34}\) respectively.

Proposition 20 The diagram
\[
\begin{array}{ccc}
\text{id}_M \otimes W_{91} & \rightarrow & M \otimes W^4_{91} \\
\downarrow & & \downarrow \\
M \otimes W^4_{(2,4),(2,4)} & \rightarrow & M \otimes W_{114} \\
\text{id}_M \otimes W_{114} & & \\
\end{array}
\]
is a pullback, where the assumptive object $E[1]$ is
$$D^7\{(2, 6), (3, 6), (4, 6), (5, 6), (1, 7), (2, 7), (3, 7), (4, 7), (5, 7), (6, 7), (2, 4), (2, 5), (3, 4), (3, 5)\},$$
the assumptive mapping
$$\eta_1^1 : D^4\{(2, 4), (3, 4)\} \to E[1]$$
is
$$(d_1, d_2, d_3, d_4) \in D^4\{(2, 4), (3, 4)\} \mapsto (d_1, d_2, d_3, 0, 0, d_4, 0) \in E[1],$$
and the assumptive mapping
$$\eta_2^1 : D^4\{(2, 4), (3, 4)\} \to E[1]$$
is
$$(d_1, d_2, d_3, d_4) \in D^4\{(2, 4), (3, 4)\} \mapsto (d_1, 0, 0, d_2, d_3, d_4, d_1d_4) \in E[1].$$

**Proof.** This follows from the microlinearity of $M$ and the pullback diagram of Weil algebras

\[
\begin{array}{ccc}
W_{E[1]} & \xrightarrow{W_{\eta_1}} & W_{D^4(2, 4, 3, 4)} \\
\downarrow W_{\eta_2} & & \downarrow W_{D^2} \\
W_{D^4(2, 4, 3, 4)} & \xrightarrow{W_{\psi_4}} & W_{D^2}
\end{array}
\]

**Notation 21** We will write $\iota_1^1$, $\iota_2^1$, $\iota_3^1$ and $\iota_4^1$ for the assumptive mappings $\eta_1^1 \circ \psi_1^3$, $\eta_2^1 \circ \psi_1^3$, $\eta_1^3 \circ \psi_3^3$ and $\eta_2^3 \circ \psi_3^3$ respectively. That is to say, we have

- $\iota_1^1 : (d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0, 0, 0, 0) \in E[1]$
- $\iota_2^1 : (d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0, 0, d_2d_3, 0) \in E[1]$
- $\iota_3^1 : (d_1, d_2, d_3) \in D^3 \mapsto (d_1, 0, 0, d_2, d_3, 0, 0) \in E[1]$
- $\iota_4^1 : (d_1, d_2, d_3) \in D^3 \mapsto (d_1, 0, 0, d_2, d_3, d_1d_2d_3) \in E[1]$

**Corollary 22** We have
\[
\left(\frac{(M \otimes W_{D^3})}{\times M \otimes W_{D^3(2, 3)}}\right) \times M \otimes W_{D^2(2)} \left(\frac{(M \otimes W_{D^3})}{\times M \otimes W_{D^3(2, 3)}}\right) = M \otimes W_{E[1]}
\]
with the diagrams

\[
\begin{array}{ccc}
\left( \begin{array}{c}
M \otimes W_{D^3} \\
\times M \otimes W_{D^3}(2,3) \\
M \otimes W_{D^3}
\end{array} \right) & \times M \otimes W_{D^3(2,3)} & \left( \begin{array}{c}
M \otimes W_{D^3} \\
\times M \otimes W_{D^3}(2,3) \\
M \otimes W_{D^3}
\end{array} \right) \\
\parallel & \parallel & \rightarrow \\
M \otimes W_{E[1]} & \rightarrow & M \otimes W_{E[1]}
\end{array}
\]

\[
\left( \begin{array}{c}
M \otimes W_{D^3} \\
\times M \otimes W_{D^3(2,3)} \\
M \otimes W_{D^3}
\end{array} \right) & \times M \otimes W_{D^3(2,3)} & \left( \begin{array}{c}
M \otimes W_{D^3} \\
\times M \otimes W_{D^3(2,3)} \\
M \otimes W_{D^3}
\end{array} \right) \\
\parallel & \parallel & \rightarrow \\
M \otimes W_{E[1]} & \rightarrow & M \otimes W_{E[1]}
\end{array}
\]

\[
\left( \begin{array}{c}
M \otimes W_{D^3} \\
\times M \otimes W_{D^3(2,3)} \\
M \otimes W_{D^3}
\end{array} \right) & \times M \otimes W_{D^3(2,3)} & \left( \begin{array}{c}
M \otimes W_{D^3} \\
\times M \otimes W_{D^3(2,3)} \\
M \otimes W_{D^3}
\end{array} \right) \\
\parallel & \parallel & \rightarrow \\
M \otimes W_{E[1]} & \rightarrow & M \otimes W_{E[1]}
\end{array}
\]

\[
\left( \begin{array}{c}
M \otimes W_{D^3} \\
\times M \otimes W_{D^3(2,3)} \\
M \otimes W_{D^3}
\end{array} \right) & \times M \otimes W_{D^3(2,3)} & \left( \begin{array}{c}
M \otimes W_{D^3} \\
\times M \otimes W_{D^3(2,3)} \\
M \otimes W_{D^3}
\end{array} \right) \\
\parallel & \parallel & \rightarrow \\
M \otimes W_{E[1]} & \rightarrow & M \otimes W_{E[1]}
\end{array}
\]

being commutative, where unnamed arrows are canonical projections.

**Proposition 23** The diagram

\[
\begin{array}{ccc}
id_M \otimes W_{\eta_1^2} & \rightarrow & id_M \otimes W_{D^4(1,4), (3,4)} \\
\downarrow & & \downarrow \\
M \otimes W_{E[2]} & \rightarrow & M \otimes W_{D^4(1,4), (3,4)} \\
\downarrow & & \downarrow \\
id_M \otimes W_{\eta_2^2} & \rightarrow & id_M \otimes W_{D^4(1,4), (3,4)} \\
\end{array}
\]

is a pullback, where the assumptive object \(E[2]\) is

\[D^7\{(1,6), (3,6), (4,6), (5,6), (1,7), (2,7), (3,7), (4,7), (5,7), (6,7), (1,4), (1,5), (3,4), (3,5)\},\]

the assumptive mapping \(\eta_1^2 : D^4\{(1,4), (3,4)\} \rightarrow E[2]\) is

\[(d_1, d_2, d_3, d_4) \in D^4\{(1,4), (3,4)\} \mapsto (d_1, d_2, d_3, 0, 0) \in E[2],\]

and the assumptive mapping \(\eta_2^2 : D^4\{(1,4), (3,4)\} \rightarrow E[2]\) is

\[(d_1, d_2, d_3, d_4) \in D^4\{(1,4), (3,4)\} \mapsto (0, d_2, 0, d_3, d_4, d_2d_4) \in E[2].\]
**Corollary 25** We have

\[ \begin{array}{ccc}
\mathcal{W}_{\eta_2^j} & \xrightarrow{\mathcal{W}_{\mathcal{E}[2]^2}} & \mathcal{W}_{\mathcal{D}^{(1,4),(3,4)}} \\
\downarrow & & \downarrow \\
\mathcal{W}_{\mathcal{D}^{(1,4),(3,4)}} & \xrightarrow{\mathcal{W}_{\mathcal{Z}_4^3}} & \mathcal{W}_{\mathcal{D}(2)}
\end{array} \]

**Notation 24** We will write \( \iota_1^2, \iota_2^2, \iota_3^2 \) and \( \iota_4^2 \) for the assumptive mappings \( \eta_1^2 \circ \varphi_2^3, \eta_2^2 \circ \psi_2^3, \varphi_2^3 \) and \( \psi_2^3 \) respectively. That is to say, we have

\[ \begin{align*}
\iota_1^2 &: (d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0, 0, 0) \in \mathcal{E}[2] \\
\iota_2^2 &: (d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0, 0, 0) \in \mathcal{E}[2] \\
\iota_3^2 &: (d_1, d_2, d_3) \in D^3 \mapsto (0, d_2, 0, d_3, d_3, 0) \in \mathcal{E}[2] \\
\iota_4^2 &: (d_1, d_2, d_3) \in D^3 \mapsto (0, d_2, 0, d_3, d_1 d_3, d_2 d_3) \in \mathcal{E}[2]
\end{align*} \]

**Corollary 25** We have

\[ \begin{array}{ccc}
(M \otimes \mathcal{W}_{D^3}) & \xrightarrow{\times \mathcal{M} \otimes \mathcal{W}_{D^3}^{(1,3)}} & (M \otimes \mathcal{W}_{D^3}) \\
\downarrow & & \downarrow \\
(M \otimes \mathcal{W}_{D^3}) & \xrightarrow{\times \mathcal{M} \otimes \mathcal{W}_{D^3}^{(1,3)}} & (M \otimes \mathcal{W}_{D^3})
\end{array} \]

with the diagrams

\[ \begin{array}{ccc}
(M \otimes \mathcal{W}_{D^3}) & \xrightarrow{\times \mathcal{M} \otimes \mathcal{W}_{D^3}^{(1,3)}} & (M \otimes \mathcal{W}_{D^3}) \\
\downarrow & & \downarrow \\
M \otimes \mathcal{W}_{\mathcal{E}[2]} & \rightarrow & \mathcal{W}_{\mathcal{D}^{(1,4),(3,4)}}
\end{array} \]

and

\[ \begin{array}{ccc}
(M \otimes \mathcal{W}_{D^3}) & \xrightarrow{\times \mathcal{M} \otimes \mathcal{W}_{D^3}^{(1,3)}} & (M \otimes \mathcal{W}_{D^3}) \\
\downarrow & & \downarrow \\
M \otimes \mathcal{W}_{\mathcal{E}[2]} & \rightarrow & \mathcal{W}_{\mathcal{D}^{(1,4),(3,4)}}
\end{array} \]

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being commutative, where unnamed arrows are the canonical projections.

**Proposition 26** The diagram

\[
\begin{array}{ccc}
M \otimes W_{E[3]} & \xrightarrow{id_M \otimes W_{\eta_1^3}} & M \otimes W_{D^4((1,4),(2,4))} \\
\downarrow & & \downarrow \\
M \otimes W_{D^4((1,4),(2,4))} & \xrightarrow{id_M \otimes W_{\eta_3^4}} & M \otimes W_{D^2(2)}
\end{array}
\]

is a pullback, where the assumptive object \(E[3]\) is 
\[D^7 \{ (1,6), (2,6), (4,6), (5,6), (1,7), (2,7), (3,7), (4,7), (5,7), (6,7), (1,4), (1,5), (2,4), (2,5) \}\),
the assumptive mapping \(\eta_1^3 : D^4 \{ (1,4), (2,4) \} \to E[3]\) is 
\[(d_1, d_2, d_3, d_4) \in D^4 \{ (1,4), (2,4) \} \mapsto (d_1, d_2, d_3, 0, 0, d_4, 0) \in E[3],\]
and the assumptive mapping \(\eta_2^3 : D^4 \{ (1,4), (3,4) \} \to E[3]\) is 
\[(d_1, d_2, d_3, d_4) \in D^4 \{ (1,4), (2,4) \} \mapsto (0, 0, d_3, d_1, d_2, d_4, d_3d_4) \in E[3].\]

**Proof.** This follows from the microlinearity of \(M\) and the pullback diagram of Weil algebras

\[
\begin{array}{ccc}
W_{\eta_2^3} & \xrightarrow{W_{E[3]}} & W_{\eta_1^3} \\
\downarrow & & \downarrow \\
W_{D^4((1,4),(2,4))} & \xrightarrow{W_{\eta_3^4}} & W_{D^2(2)}
\end{array}
\]

**Notation 27** We will write \(i_1^3, i_2^3, i_3^3, i_4^3\) and \(i_5^3\) for the assumptive mappings \(\eta_1^3 \circ \varphi_3^3, \eta_1^3 \circ \psi_3^3, \eta_2^3 \circ \varphi_3^3\) and \(\eta_3^3 \circ \psi_3^3\) respectively. That is to say, we have
\[
\begin{align*}
i_1^3 &: (d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0, 0, 0, 0) \in E[3] \\
i_2^3 &: (d_1, d_2, d_3) \in D^3 \mapsto (d_1, d_2, d_3, 0, 0, d_1d_2, 0) \in E[3] \\
i_3^3 &: (d_1, d_2, d_3) \in D^3 \mapsto (0, 0, d_3, d_1, d_2, 0, d_3d_4) \in E[3] \\
i_4^3 &: (d_1, d_2, d_3) \in D^3 \mapsto (0, 0, d_3, d_1, d_2, d_1d_2d_3) \in E[3]
\end{align*}
\]

**Corollary 28** We have
\[
\begin{pmatrix}
(M \otimes W_{D^3}) \\
\times M \otimes W_{D^3((1,2))}
\end{pmatrix}
\times
\begin{pmatrix}
(M \otimes W_{D^3}) \\
\times M \otimes W_{D^3((1,2))}
\end{pmatrix}
= M \otimes W_{E[3]}
\]
with the diagrams

\[
\begin{pmatrix}
(M \otimes W_{D^3})^1 \\
\times M \otimes W_{D^3(1,2)}
\end{pmatrix}
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3})^2
\]
\[\longrightarrow\]

\[
\begin{pmatrix}
(M \otimes W_{D^3})^1 \\
\times M \otimes W_{D^3(1,2)}
\end{pmatrix}
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3})^2
\]
\[\longrightarrow\]

\[
M \otimes W_{E[3]} \times M \otimes W_{E[3]} \rightarrow id_M \otimes W_{\eta_2^3}
\]

and

\[
\begin{pmatrix}
(M \otimes W_{D^3})^1 \\
\times M \otimes W_{D^3(1,2)}
\end{pmatrix}
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3})^2
\]
\[\longrightarrow\]

\[
\begin{pmatrix}
(M \otimes W_{D^3})^1 \\
\times M \otimes W_{D^3(1,2)}
\end{pmatrix}
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3})^2
\]
\[\longrightarrow\]

\[
M \otimes W_{E[3]} \times M \otimes W_{E[3]} \rightarrow id_M \otimes W_{\eta_2^3}
\]

being commutative, where unnamed arrows are the canonical projections.

Now we come to the crucial step in the proof of the general Jacobi identity.
Notation 29  A limit of the diagram

\[
\begin{array}{cc}
(M \otimes W_{D^3})_{321} \
\times M \otimes W_{D^3(2,3)} \
(M \otimes W_{D^3})_{231} \\
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3})_{312} \\
\times M \otimes W_{D^3(1,3)} \\
(M \otimes W_{D^3})_{213} \\
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3})_{231} \\
\end{array}
\]

\[
\begin{array}{cc}
(M \otimes W_{D^3})_{123} \\
\times M \\
(M \otimes W_{D^3})_{321} \\
\times M \otimes W_{D^3(2,3)} \\
(M \otimes W_{D^3})_{231} \\
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3})_{312} \\
\times M \otimes W_{D^3(1,3)} \\
(M \otimes W_{D^3})_{213} \\
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3})_{231} \\
\end{array}
\]
with every arrow being the canonical projection is denoted by

\[
\begin{bmatrix}
(M \otimes W_{D^3}^{1})_{321} & (M \otimes W_{D^3}^{1})_{231} & (M \otimes W_{D^3}^{1})_{132} & (M \otimes W_{D^3}^{1})_{123} \\
(M \otimes W_{D^3}^{3, (1, 3)})_{312} & (M \otimes W_{D^3}^{3, (1, 2)})_{213} & (M \otimes W_{D^3}^{3, (1, 2)})_{231} & (M \otimes W_{D^3}^{3, (1, 2)})_{123} \\
(M \otimes W_{D^3}^{3, (1, 3)})_{132} & (M \otimes W_{D^3}^{3, (1, 2)})_{312} & (M \otimes W_{D^3}^{3, (1, 2)})_{231} & (M \otimes W_{D^3}^{3, (1, 2)})_{123} \\
(M \otimes W_{D^3}^{3, (1, 3)})_{231} & (M \otimes W_{D^3}^{3, (1, 2)})_{123} & (M \otimes W_{D^3}^{3, (1, 2)})_{231} & (M \otimes W_{D^3}^{3, (1, 2)})_{123} \\
\end{bmatrix}
\]

We can compute the above limit.

**Theorem 30** The diagram

\[
\begin{array}{cccc}
\text{id}_M \otimes W_{k_2} & M \otimes W_{E[1]} & \text{id}_M \otimes W_{k_1} \\
\text{id}_M \otimes W_{k_1} & M \otimes W_G & \text{id}_M \otimes W_{E[2]} \\
\text{id}_M \otimes W_{k_2} & M \otimes W_{D^3 \otimes D^3} & \text{id}_M \otimes W_{k_3} \\
\end{array}
\]

is a limit diagram with the three unnamed arrows being

\[
\begin{array}{c}
\text{id}_M \otimes W_{k_1} : M \otimes W_G \rightarrow M \otimes W_{E[1]} \\
\text{id}_M \otimes W_{k_2} : M \otimes W_G \rightarrow M \otimes W_{E[2]} \\
\text{id}_M \otimes W_{k_3} : M \otimes W_G \rightarrow M \otimes W_{E[3]} \\
\end{array}
\]

where the assumptive object \( G \) is

\[
D^8 \{ (2, 4), (3, 4), (1, 5), (3, 5), (1, 6), (2, 6), (4, 5), (4, 6), (5, 6), (1, 7), (2, 7), (3, 7), (4, 7), (5, 7), (6, 7), (1, 8), (2, 8), (3, 8), (4, 8), (5, 8), (6, 8), (7, 8) \},
\]
the assumptive mapping $k_1 : E[1] \rightarrow G$ is

$$(d_1, d_2, d_3, d_4, d_5, d_6) \in E[1] \mapsto (d_1, d_2 + d_4, d_3 + d_5, d_6 - d_2d_3 - d_4d_5, -d_1d_5, d_1d_4, d_1d_2d_3, d_1d_2d_4) \in G,$$

the assumptive mapping $k_2 : E[2] \rightarrow G$ is

$$(d_1, d_2, d_3, d_4, d_5, d_6) \in E[2] \mapsto (d_1 + d_5, d_2, d_3 + d_4, -d_2d_3, d_6 - d_1d_3 - d_4d_5, d_1d_2, d_2d_4d_5, d_7) \in G,$$

the assumptive mapping $k_3 : E[3] \rightarrow G$ is

$$(d_1, d_2, d_3, d_4, d_5, d_6) \in E[3] \mapsto (d_1 + d_4, d_2 + d_5, d_3, -d_4d_5, -d_1d_3, d_6, -d_7, -d_7 + d_3d_4d_5) \in G,$$

the assumptive mapping $h_{12}^1$ is

$$\iota_2^1 \oplus \iota_3^1,$$

the assumptive mapping $h_{12}^2$ is

$$\iota_2^2 \oplus \iota_3^2,$$

the assumptive mapping $h_{23}^2$ is

$$\iota_2^2 \oplus \iota_3^3,$$

the assumptive mapping $h_{23}^3$ is

$$\iota_3^3 \oplus \iota_3^3,$$

the assumptive mapping $h_{31}^3$ is

$$\iota_3^3 \oplus \iota_3^3,$$

and the assumptive mapping $h_{31}^4$ is

$$\iota_3^4 \oplus \iota_3^4.$$
Corollary 31. We have

\[
\begin{array}{c}
\begin{pmatrix}
(M \otimes W_D^3)_{321} \\
\times M \otimes W_D^{(1,3)}_{(2,3)} \\
(M \otimes W_D^3)_{231} \\
\times M \otimes W_D^{(2)} \\
(M \otimes W_D^3)_{132} \\
\times M \otimes W_D^{(1,3)}_{(1,2)} \\
(M \otimes W_D^3)_{123} \\
\times M \otimes W_D^{(2)} \\
(M \otimes W_D^3)_{231} \\
\times M \otimes W_D^{(1,3)}_{(1,2)} \\
(M \otimes W_D^3)_{312} \\
\times M \otimes W_D^{(2)} \\
(M \otimes W_D^3)_{312} \\
\end{pmatrix}
\end{array}
\times
\begin{array}{c}
\begin{pmatrix}
(M \otimes W_D^3)_{132} \\
\times M \otimes W_D^{(1,3)}_{(1,2)} \\
(M \otimes W_D^3)_{231} \\
\times M \otimes W_D^{(2)} \\
(M \otimes W_D^3)_{123} \\
\times M \otimes W_D^{(1,3)}_{(1,2)} \\
(M \otimes W_D^3)_{312} \\
\times M \otimes W_D^{(2)} \\
(M \otimes W_D^3)_{312} \\
\end{pmatrix}
\end{array}
\]

= M \otimes W_G

with the diagrams:

\[
\begin{array}{c}
\begin{pmatrix}
(M \otimes W_D^3)_{132} \\
\times M \otimes W_D^{(1,3)} \\
(M \otimes W_D^3)_{312} \\
\times M \otimes W_D^{(2)} \\
(M \otimes W_D^3)_{213} \\
\times M \otimes W_D^{(1,2)} \\
(M \otimes W_D^3)_{231} \\
\times M \otimes W_D^{(2)} \\
(M \otimes W_D^3)_{312} \\
\end{pmatrix}
\end{array}
\triangle
\rightarrow
\begin{array}{c}
\begin{pmatrix}
(M \otimes W_D^3)_{132} \\
\times M \otimes W_D^{(1,3)} \\
(M \otimes W_D^3)_{312} \\
\times M \otimes W_D^{(2)} \\
(M \otimes W_D^3)_{213} \\
\times M \otimes W_D^{(1,2)} \\
(M \otimes W_D^3)_{231} \\
\times M \otimes W_D^{(2)} \\
(M \otimes W_D^3)_{312} \\
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{c}
\begin{pmatrix}
M \otimes W_G \\
\text{id}_M \otimes W_k \\
M \otimes W_{E[2]} \\
\end{pmatrix}
\end{array}
\rightarrow
\begin{array}{c}
\begin{pmatrix}
M \otimes W_G \\
\text{id}_M \otimes W_k \\
M \otimes W_{E[2]} \\
\end{pmatrix}
\end{array}
\]
and unnamed arrows are the canonical projections.

The proof of the above theorem follows directly from the following lemma.
Lemma 32 The following diagram is a limit diagram of Weil algebras:

Proof. Let \( \gamma_1 \in \mathcal{W}_{E[1]} \), \( \gamma_2 \in \mathcal{W}_{E[2]} \), \( \gamma_3 \in \mathcal{W}_{E[3]} \) and \( \gamma \in \mathcal{W}_G \) so that they are polynomials with coefficients in \( k \) of the following forms:

\[
\gamma_1(X_1, X_2, X_3, X_4, X_5, X_6, X_7) = a^1 + a_1^1 X_1 + a_2^1 X_2 + a_3^1 X_3 + a_4^1 X_4 + a_5^1 X_5 + a_6^1 X_6 + a_7^1 X_7 + a_{12}^1 X_1 X_2 + a_{13}^1 X_1 X_3 + a_{14}^1 X_1 X_4 + a_{15}^1 X_1 X_5 + a_{16}^1 X_1 X_6 + a_{23}^1 X_2 X_3 + a_{15}^1 X_4 X_5 + a_{123}^1 X_1 X_2 X_3 + a_{145}^1 X_1 X_4 X_5
\]

\[
\gamma_2(X_1, X_2, X_3, X_4, X_5, X_6, X_7) = a^2 + a_1^2 X_1 + a_2^2 X_2 + a_3^2 X_3 + a_4^2 X_4 + a_5^2 X_5 + a_6^2 X_6 + a_7^2 X_7 + a_{12}^2 X_1 X_2 + a_{13}^2 X_1 X_3 + a_{23}^2 X_2 X_3 + a_{24}^2 X_2 X_4 + a_{25}^2 X_2 X_5 + a_{26}^2 X_2 X_6 + a_{27}^2 X_2 X_7 + a_{123}^2 X_1 X_2 X_3 + a_{245}^2 X_2 X_4 X_5
\]

\[
\gamma_3(X_1, X_2, X_3, X_4, X_5, X_6, X_7) = a^3 + a_1^3 X_1 + a_2^3 X_2 + a_3^3 X_3 + a_4^3 X_4 + a_5^3 X_5 + a_6^3 X_6 + a_7^3 X_7 + a_{12}^3 X_1 X_2 + a_{13}^3 X_1 X_3 + a_{23}^3 X_2 X_3 + a_{24}^3 X_2 X_4 + a_{35}^3 X_3 X_5 + a_{36}^3 X_3 X_6 + a_{37}^3 X_3 X_7 + a_{123}^3 X_1 X_2 X_3 + a_{345}^3 X_3 X_4 X_5
\]

\[
\gamma(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = b + b_1 X_1 + b_2 X_2 + b_3 X_3 + b_4 X_4 + b_5 X_5 + b_6 X_6 + b_7 X_7 + b_8 X_8 + b_{12} X_1 X_2 + b_{13} X_1 X_3 + b_{14} X_1 X_4 + b_{23} X_2 X_3 + b_{25} X_2 X_5 + b_{36} X_3 X_6
\]

It is easy to see that

\[
\mathcal{W}_{E[1]}(\gamma_1)(X_1, X_2, X_3, X_4, X_5, X_6) = a^1 + a_1^1 X_1 + a_2^1 X_2 + a_3^1 X_3 + a_4^1 X_4 + a_5^1 X_5 + a_6^1 X_6 + a_7^1 X_7 + a_{12}^1 X_1 X_2 + a_{13}^1 X_1 X_3 + a_{14}^1 X_1 X_4 + a_{15}^1 X_1 X_5 + a_{16}^1 X_1 X_6 + a_{23}^1 X_2 X_3
\]

\[
+ a_{123}^1 X_1 X_2 X_3 + a_{145}^1 X_4 X_5 + a_{124}^1 X_4 X_5 + a_{235}^1 X_2 X_3 + a_{135}^1 X_1 X_5 + a_{145}^1 X_4 X_5 + a_{15}^1 X_4 X_5 + a_{16}^1 X_4 X_5 + a_{23}^1 X_2 X_3
\]

\[
+ a_{123}^1 X_1 X_2 X_3 + a_{145}^1 X_4 X_5 + a_{124}^1 X_4 X_5 + a_{235}^1 X_2 X_3 + a_{135}^1 X_1 X_5 + a_{145}^1 X_4 X_5 + a_{15}^1 X_4 X_5 + a_{16}^1 X_4 X_5 + a_{23}^1 X_2 X_3
\]

\[
+ a_{123}^1 X_1 X_2 X_3 + a_{145}^1 X_4 X_5 + a_{124}^1 X_4 X_5 + a_{235}^1 X_2 X_3 + a_{135}^1 X_1 X_5 + a_{145}^1 X_4 X_5 + a_{15}^1 X_4 X_5 + a_{16}^1 X_4 X_5 + a_{23}^1 X_2 X_3
\]

\[
+ a_{123}^1 X_1 X_2 X_3 + a_{145}^1 X_4 X_5 + a_{124}^1 X_4 X_5 + a_{235}^1 X_2 X_3 + a_{135}^1 X_1 X_5 + a_{145}^1 X_4 X_5 + a_{15}^1 X_4 X_5 + a_{16}^1 X_4 X_5 + a_{23}^1 X_2 X_3
\]

\[
+ a_{123}^1 X_1 X_2 X_3 + a_{145}^1 X_4 X_5 + a_{124}^1 X_4 X_5 + a_{235}^1 X_2 X_3 + a_{135}^1 X_1 X_5 + a_{145}^1 X_4 X_5 + a_{15}^1 X_4 X_5 + a_{16}^1 X_4 X_5 + a_{23}^1 X_2 X_3
\]

22
\[ W_{h_{12}}(\gamma_2)(X_1, X_2, X_3, X_4, X_5, X_6) \]
\[ = a^2 + a_2^2 X_2 + a_3^2 X_3 + a_4^2 X_1 + a_5^2 X_1 X_3 + a_6^2 X_1 X_2 X_3 + a_{24}^2 X_2 X_3 + a_{25}^2 X_1 X_2 + a_{26}^2 X_1 X_2 X_3 + a_{45}^2 X_1 X_2 X_3 + a_7^2 X_4 + a_8^2 X_5 + a_9^2 X_6 + a_{12}^2 X_4 X_5 + a_{13}^2 X_4 X_6 + a_{23}^2 X_5 X_6 + a_{123}^2 X_4 X_5 X_6 \]
\[ = a^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2 + a_9^2 + a_{12}^2 + a_{15}^2 + a_{16}^2 + a_{12,3}^2 + a_{123}^2 \]

Therefore the condition that \( W_{h_{12}}(\gamma_1) = W_{h_{12}}(\gamma_2) \) is equivalent to the following conditions as a whole:

\[ a^1 = a^2 \] (1)
\[ a_1^1 = a_2^2, a_1^1 = a_2^2, a_1^1 = a_2^2, a_1^1 = a_2^2, a_1^1 = a_2^2 \] (2)
\[ a_{12}^1 = a_{25}^2, a_{13}^1 = a_6^2 + a_{1,2}^2, a_6^2 + a_{13}^2 = a_{24}^2, a_{14}^1 = a_{12}^1, a_{15}^1 = a_{13}^2 \] (3)
\[ a_{16}^1 + a_{123}^1 = a_7^2 + a_{26}^2 + a_{245}^2, a_{145}^1 = a_{123}^2 \] (4)

By the same token, the condition that \( W_{h_{23}}(\gamma_2) = W_{h_{23}}(\gamma_3) \) is equivalent to the following conditions as a whole:

\[ a^2 = a^3 \] (5)
\[ a_2^2 = a_3^3, a_2^2 = a_3^3, a_2^2 = a_3^3, a_2^2 = a_3^3, a_2^2 = a_3^3 \] (6)
\[ a_{23}^2 = a_{35}^3, a_{12}^2 = a_6^3 + a_{145}^2, a_6^3 + a_{13}^2 = a_{34}^3, a_{24}^2 = a_{23}^2, a_{25}^2 = a_{12}^3 \] (7)
\[ a_{26}^2 + a_{123}^2 = a_7^3 + a_{36}^3 + a_{245}^3, a_{245}^2 = a_{123}^3 \] (8)

By the same token again, the condition that \( W_{h_{31}}(\gamma_3) = W_{h_{31}}(\gamma_1) \) is equivalent to the following conditions as a whole:

\[ a^3 = a^1 \] (9)
\[ a_3^3 = a_7^1, a_3^3 = a_1^1, a_2^3 = a_4^1, a_3^3 = a_1^1, a_2^3 = a_4^1, a_3^3 = a_1^1, a_2^3 = a_4^1, a_3^3 = a_1^1, a_2^3 = a_4^1 \] (10)
\[ a_{13}^3 = a_{15}^4, a_{23}^3 = a_{6}^4 + a_{145}^1, a_6^3 + a_{13}^2 = a_{14}^1, a_{34}^3 = a_{13}^1, a_{35}^3 = a_{23}^1 \] (11)
\[ a_{36}^3 + a_{123}^3 = a_7^1 + a_{16}^1 + a_{145}^1, a_{345}^3 = a_{123}^1 \] (12)

The three conditions (1), (5) and (9) can be combined into

\[ a^1 = a^2 = a^3 \] (13)
The three conditions (2), (6) and (10) are to be superseded by the following three conditions as a whole:

\[
a_1^3 = a_2^3 = a_3^3 = a_4^3 = a_5^3 = a_6^3
\]

The three conditions (3), (7) and (11) are equivalent to the following six conditions as a whole:

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

The three conditions (8), (12) and (18) are equivalent to the following six conditions as a whole:

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

The conditions (14), (18) and (22) imply that

\[
a_1^1 + a_2^2 + a_3^3 = (a_3^3 + a_1^3 - a_1^1 - a_1^2) + (a_1^1 + a_2^1 - a_2^2 - a_2^2) + (a_2^2 + a_2^2 - a_2^2 - a_2^2)
\]

\[
= (a_3^3 + a_1^3 - a_1^1 - a_1^2) + (a_1^1 + a_2^1 - a_2^2 - a_2^2) + (a_2^2 + a_2^2 - a_2^2 - a_2^2)
\]

\[
= 0
\]

Therefore the three conditions (4), (8) and (12) are to be replaced by the following five conditions as a whole:

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

\[
a_1^1 = a_2^1 = a_3^1 = a_4^1 = a_5^2 = a_6^2
\]

Indeed, the condition that \(a_3^3 = a_1^3\) is derivable from the above five conditions, as is to be demonstrated in the following:

\[
a_3^3 = a_3^3 + a_3^3 - a_3^3
\]

\[
a_3^3 = a_3^3 + a_3^3 - a_3^3
\]

\[
a_3^3 = a_3^3 + a_3^3 - a_3^3
\]

\[
a_1^1 + a_2^2 + a_3^3 = a_1^1 + a_2^2 + a_3^3
\]

\[
a_1^1 + a_2^2 + a_3^3 = a_1^1 + a_2^2 + a_3^3
\]

\[
a_1^1 + a_2^2 + a_3^3 = a_1^1 + a_2^2 + a_3^3
\]
Now it is not difficult to see that $W_{h_i^1}(\gamma_1) = W_{h_i^2}(\gamma_2)$, $W_{h_i^2}(\gamma_2) = W_{h_i^3}(\gamma_3)$ and $W_{h_i^3}(\gamma_3) = W_{h_i^1}(\gamma_1)$ exactly when there exists $\gamma \in W_G$ with $\gamma_i = W_{h_i}(\gamma)$ ($i = 1, 2, 3$), in which $\gamma$ should uniquely be of the following form:

$$
\begin{align*}
\gamma(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) &= a^1 + a_1^1 X_1 + a_2^1 X_2 + a_3^1 X_3 + a_4^1 X_4 + a_5^1 X_5 + a_6^1 X_6 + a_7^1 X_7 + a_8^1 X_8 + a_{12}^1 X_1 X_2 \\
&+ a_{13}^1 X_1 X_3 + a_{16}^1 X_1 X_4 + (a_{23}^1 + a_{6}^1) X_2 X_3 + a_{26}^1 X_2 X_5 + a_{36}^1 X_3 X_6
\end{align*}
$$

This completes the proof of the theorem. ■

**Notation 33** We will introduce three notations.

1. We will write

$$
\zeta\left(\ast_{123}^1 - \ast_{132}^1 - \ast_{213}^1\right):
$$

$$
\begin{bmatrix}
(M \otimes W_{D^3})_{321} \\
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3})_{231} \\
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3})_{132} \\
\times M \otimes W_{D^3(1,2)} \\
(M \otimes W_{D^3(1,2)})_{123} \\
\times M \otimes W_{D^3(1,2)} \\
\times M \otimes W_{D^3(1,2)}
\end{bmatrix}
\rightarrow M \otimes W_D
$$
for the composition of morphisms

\[
\pi^\triangle \left( \left( \left( M \otimes_W D^3 \right)_{321} \times_{M \otimes_W D^3_{(2,3)}} \left( M \otimes_W D^3 \right)_{231} \right) \times_{M \otimes_W D^3_{(2,3)}} \left( M \otimes_W D^3 \right)_{132} \right) \times_{M \otimes_W D^3_{(2,3)}} \left( M \otimes_W D^3 \right)_{123} \right):
\]

\[
\begin{array}{c}
(M \otimes W_{D^3})_{321} \\
\times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{231} \\
\times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{132} \\
\times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{123} \\
\end{array}
\]

\[
(M \otimes W^3)_{132} \\
\times_{M \otimes W_{D^3_{(1,3)}}} (M \otimes W_{D^3})_{312} \\
\times_{M \otimes W_{D^3_{(1,3)}}} (M \otimes W_{D^3})_{132} \\
\times_{M \otimes W_{D^3_{(1,3)}}} (M \otimes W_{D^3})_{123} \\
\]

\[
\rightarrow (M \otimes W^3)_{321} \\
\times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{231} \\
\times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{132} \\
\times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{123} \\
\]

\[
\zeta^+ \times_{M \otimes W_{D^3_{(2,3)}}} \zeta^+ : \\
\left( (M \otimes W^3)_{321} \right) \\
\times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{231} \\
\times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{132} \\
\times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{123} \\
\rightarrow (M \otimes W^3)_{321} \times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{231} \times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{132} \times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W_{D^3})_{123} \\
\]

\[
\zeta^- : (M \otimes W^2) \times_{M \otimes W_{D^3_{(2,3)}}} (M \otimes W^2) \rightarrow M \otimes W^2 \\
in succession.
\]
2. We will write the morphism

\[
\zeta^{\binom{231 \cdot 2}{*} 231 \cdot 2 \binom{*}{*} 213 \cdot 2 * 213} : \\
\begin{pmatrix}
\begin{pmatrix}
(M \otimes W_D^{3})_{321} \\
\times M \otimes W_{D^3(1,2)}^{3(1,2)} \\
(M \otimes W_D^{3})_{132} \\
\times M \otimes W_{D^3(1,2)}^{3(1,2)} \\
(M \otimes W_D^{3})_{231} \\
\times M \otimes W_{D^3(1,2)}^{3(1,2)} \\
\end{pmatrix} \\
\times M \otimes W_{D^3(1,2)}^{3(1,2)} \\
\times M \otimes W_D^{3} \\
\end{pmatrix}
\rightarrow M \otimes W_D
\]
for the composition of morphisms

\[
\pi^\triangle \left( (M \otimes W_{D^3})_{132} \times_{M \otimes W_{D^3}^{(1,3)}} (M \otimes W_{D^3})_{312} \right) \times_{M \otimes W_{D^3}(2)} (M \otimes W_{D^3})_{213} \times_{M \otimes W_{D^3}^{(1,3)}} (M \otimes W_{D^3})_{231} \right) :
\]

\[
\times_{M \otimes W_{D^3}^{(2,3)}} (M \otimes W_{D^3})_{132} \times_{M \otimes W_{D^3}^{(2,3)}} (M \otimes W_{D^3})_{231} \times_{M \otimes W_{D^3}^{(2,3)}} (M \otimes W_{D^3})_{123} \times_{M \otimes W_{D^3}^{(2,3)}} (M \otimes W_{D^3})_{231}
\]

\[
\rightarrow (M \otimes W_{D^3})_{132} \times_{M \otimes W_{D^3}^{(1,3)}} (M \otimes W_{D^3})_{312} \times_{M \otimes W_{D^3}(2)} (M \otimes W_{D^3})_{213} \times_{M \otimes W_{D^3}^{(1,3)}} (M \otimes W_{D^3})_{231}
\]

\[
\zeta^2 \times_{M \otimes W_{D^3}(2)} \zeta^2 : \\
\left( (M \otimes W_{D^3})_{132} \times_{M \otimes W_{D^3}^{(1,3)}} (M \otimes W_{D^3})_{312} \times_{M \otimes W_{D^3}(2)} (M \otimes W_{D^3})_{213} \times_{M \otimes W_{D^3}^{(1,3)}} (M \otimes W_{D^3})_{231} \right) \\
\rightarrow (M \otimes W_{D^2}) \times_{M \otimes W_{D^3}(2)} (M \otimes W_{D^3})
\]

\[
\zeta^- : (M \otimes W_{D^2}) \times_{M \otimes W_{D^3}(2)} (M \otimes W_{D^3}) \rightarrow M \otimes W_D
\]
in succession.
3. We will write the morphism

\[ \zeta^{(312 
\begin{array}{c}
(M \otimes W_{D^3}) \\
(123 \\
\end{array}} \\
\begin{array}{c}
(M \otimes W_{D^3}) \\
(123 \\
\end{array}} \]

\[ \otimes W_{D^3} \rightarrow M \otimes W_D \]
for the composition of morphisms

\[
\pi^\Delta \left( (M \otimes W_{D^3})_{213} \times_{M \otimes W_{D^3}^{(1,3)}} (M \otimes W_{D^3})_{213} \right) \times_{M \otimes W_{D^3}^{(1,3)}} \left( (M \otimes W_{D^3})_{321} \times_{M \otimes W_{D^3}^{(2,3)}} (M \otimes W_{D^3})_{321} \right)
\]

\[
\rightarrow (M \otimes W_{D^3})_{213} \times_{M \otimes W_{D^3}^{(1,2)}} (M \otimes W_{D^3})_{213} \times_{M \otimes W_{D^3}^{(1,2)}} (M \otimes W_{D^3})_{321} \times_{M \otimes W_{D^3}^{(2,3)}} (M \otimes W_{D^3})_{321}
\]

\[
\zeta^3 \times_{M \otimes W_{D^3}^{(1,2)}} \zeta^3
\]

\[
\rightarrow (M \otimes W_{D^2}) \times_{M \otimes W_{D^3}^{(1,2)}} (M \otimes W_{D^2}) \times_{M \otimes W_{D^3}^{(2,3)}} (M \otimes W_{D^3})
\]

\[
\zeta^- : (M \otimes W_{D^2}) \times_{M \otimes W_{D^3}^{(1,2)}} (M \otimes W_{D^2}) \rightarrow M \otimes W_D
\]

in succession.
Theorem 34 \((The \ general \ Jacobi \ Identity)\) \(The \ three \ morphisms\)

\[
\zeta^{(1,2,3)}_{1} \cdot \zeta^{-(1,2,3)}_{1} \cdot \zeta^{-(2,1,3)}_{1}:
\begin{align*}
&\left(\begin{array}{c}
(M \otimes W_{D^3})_{321} \\
\times_{M \otimes W_{D^3}^{(2,3)}} \\
(M \otimes W_{D^3})_{231} \\
\times_{M \otimes W_{D^3}^{(1,2)}} \\
(M \otimes W_{D^3})_{132} \\
\times_{M \otimes W_{D^3}^{(1,3)}} \\
(M \otimes W_{D^3})_{213} \\
\times_{M \otimes W_{D^3}^{(1,2)}} \\
(M \otimes W_{D^3})_{312} \\
\end{array}\right) \\
\end{align*}
\]

\[
\rightarrow M \otimes W_{D}
\]
\[
\zeta^{(\ast_{231}\ast_{213})-(\ast_{312}\ast_{132})}:
\begin{array}{c}
\begin{bmatrix}
(M \otimes W_{D^3})_{321} \\
\times M \otimes W_{D^3, (2,3)} \\
(M \otimes W_{D^3})_{231} \\
\times M \otimes W_{D^{(2)}} \\
(M \otimes W_{D^3})_{132} \\
\times M \otimes W_{D^3, (2,3)} \\
(M \otimes W_{D^3})_{123}
\end{bmatrix}
\end{array}
\rightarrow M \otimes W_{D}
\]
\[ \zeta^{(312^{-3},321^{-3}) - (123^{-3},213^{-3})} : \]

\[
\begin{pmatrix}
(M \otimes W_{D^3})_{321} \\
X_{M \otimes W_{D^3 (2,3)}} (M \otimes W_{D^3})_{231} \\
X_{M \otimes W_{D^3 (2)}} (M \otimes W_{D^3})_{132} \\
X_{M \otimes W_{D^3 (1,3)}} (M \otimes W_{D^3})_{312} \\
X_{M \otimes W_{D^3 (1,3)}} (M \otimes W_{D^3})_{213} \\
X_{M \otimes W_{D^3 (1,3)}} (M \otimes W_{D^3})_{213}
\end{pmatrix}
\times
\begin{pmatrix}
(M \otimes W_{D^3})_{321} \\
X_{M \otimes W_{D^3 (2,3)}} (M \otimes W_{D^3})_{231} \\
X_{M \otimes W_{D^3 (2)}} (M \otimes W_{D^3})_{132} \\
X_{M \otimes W_{D^3 (1,3)}} (M \otimes W_{D^3})_{312} \\
X_{M \otimes W_{D^3 (1,3)}} (M \otimes W_{D^3})_{213} \\
X_{M \otimes W_{D^3 (1,3)}} (M \otimes W_{D^3})_{213}
\end{pmatrix}
\]

\[ \rightarrow M \otimes W_{D^3} \]

*sum up only to vanish.*

**Proof.** The proof is divided into four steps.
1. The morphism

$$\zeta^{(*_{123} \cdot *_{132}) - (*_{231} \cdot *_{232})} :$$

$$\begin{bmatrix}
(M \otimes W_{D^3})_{321} \\
\times_M \otimes W_{D^3((1,3))} \\
(M \otimes W_{D^3})_{132} \\
\times_M \otimes W_{D^3((1,2))} \\
(M \otimes W_{D^3})_{213} \\
\times_M \otimes W_{D^3((1,3))} \\
(M \otimes W_{D^3})_{231} \\
\times_M \otimes W_{D^3_{D^3} D^3}
\end{bmatrix}

\rightarrow M \otimes W_D$$

is equivalent to the composition of

$$\text{id}_{M} \otimes k_1 : M \otimes W_G \rightarrow M \otimes W_{E[1]}$$

$$\text{id}_{M} \otimes W_{d \in D^3 \rightarrow (d_1,0,0,0,d_2,0,d_3) \in E[1]} : M \otimes W_{E[1]} \rightarrow M \otimes W_{D^3_{\{1,3,2,3\}}}$$

$$\text{id}_{M} \otimes W_{d \in D^3 \rightarrow (0,0,0,0,0,d,0) \in E[1]} : M \otimes W_{D^3\{1,3,2,3\}} \rightarrow M \otimes W_D$$

in succession, which results in

$$\text{id}_{M} \otimes W_{d \in D^3 \rightarrow (0,0,0,0,0,0,d,0) \in E[1]} : M \otimes W_G \rightarrow M \otimes W_D$$
2. The morphism

\[ \zeta^{(\ast_{231}, \ast_{213}) \ast (\ast_{312}, \ast_{132})} : \]

\[
\begin{pmatrix}
(M \otimes W_{D^3})_{321} \\
\times M \otimes W_{D^3}^{(1,2,3)} \\
(M \otimes W_{D^3})_{231} \\
\times M \otimes W_{D^3}^{(1,2,3)} \\
(M \otimes W_{D^3})_{132} \\
\times M \otimes W_{D^3}^{(1,2,3)} \\
(M \otimes W_{D^3})_{213} \\
\times M \otimes W_{D^3}^{(1,2,3)} \\
(M \otimes W_{D^3})_{312} \\
\times M \otimes W_{D^3}^{(1,2,3)} \\
(M \otimes W_{D^3})_{231} \\
\times M \otimes W_{D^3}^{(1,2,3)} \\
\end{pmatrix} \]

\[\rightarrow M \otimes W_D\]

is equivalent to the composition of

\[\text{id}_M \otimes k_2 : M \otimes W_G \rightarrow M \otimes W_E^{[2]}\]

\[\text{id}_M \otimes W_{(d_1, d_2, d_3, d_4) \in D^3 \{ (1,3), (2,3) \}} \rightarrow (0, d_1, 0, 0, 0, d_2, d_3, d_4) \in E[2] : M \otimes W_E^{[2]} \rightarrow M \otimes W_{D^3 \{ (1,3), (2,3) \}}\]

\[\text{id}_M \otimes W_{d \in D \rightarrow (0,0,0,0,0,0,0, d) \in G} : M \otimes W_{D^3 \{ (1,3), (2,3) \}} \rightarrow M \otimes W_D\]

in succession, which results in

\[\text{id}_M \otimes W_{d \in D \rightarrow (0,0,0,0,0,0,0, d) \in G} : M \otimes W_G \rightarrow M \otimes W_D\]
3. The morphism

\[ \zeta = (\ast_{312} \ast_{321}) - (\ast_{123} \ast_{213}) : \]

\[
\begin{pmatrix}
(M \otimes W_{D^3})_{321} \\
\times M \otimes W_{D^3 (1,2)} \\
(M \otimes W_{D^3})_{231} \\
\times M \otimes W_{D^3 (2,3)} \\
(M \otimes W_{D^3})_{132} \\
\times M \otimes W_{D^3 (1,3)} \\
(M \otimes W_{D^3})_{213} \\
\times M \otimes W_{D^3 (1,1)} \\
(M \otimes W_{D^3})_{231}
\end{pmatrix}
\]

\[
\rightarrow M \otimes W_D
\]

is equivalent to the composition of

\[ \text{id}_M \otimes k_3 : M \otimes W_G \rightarrow M \otimes W_{E[3]} \]

\[ \text{id}_M \otimes W_{d_1,d_2,d_3} : M \otimes W_{d_1,d_2,d_3} \rightarrow M \otimes W_{D^3 (1,1), (2,3)} \]

\[ \text{id}_M \otimes W_{d} : M \otimes W_{D^3 (1,1), (2,3)} \rightarrow M \otimes W_D \]

in succession, which results in

\[ \text{id}_M \otimes W_{d_1,d_2,d_3} : M \otimes W_G \rightarrow M \otimes W_D \]
4. Therefore

\[ \zeta^{(123^1 - 1^2)} + \zeta^{(123^2 - 1^3)} + \zeta^{(132^2 - 1^3)} :\]

\[
\begin{bmatrix}
(M \otimes W_{D^3})_{321} \\
\times M \otimes W_{D^3, (2,3)} \\
(M \otimes W_{D^3})_{231} \\
\times M \otimes W_{D^3, (2,1)} \\
(M \otimes W_{D^3})_{132} \\
\times M \otimes W_{D^3, (1,2)} \\
(M \otimes W_{D^3})_{312} \\
\times M \otimes W_{D^3, (1,3)} \\
(M \otimes W_{D^3})_{213} \\
\times M \otimes W_{D^3, (3,1)} \\
(M \otimes W_{D^3})_{123} \\
\times M \otimes W_{D^3, (3,2)} \\
\end{bmatrix}
\]

\[\to M \otimes W_D\]

is equivalent to

\[
(id_M \otimes W_{d \in D \rightarrow (d,d,d) \in D(3)}) \circ (id_M \otimes W_{(d_1,d_2,d_3) \in D(3) \rightarrow (0,0,0,0,0,0,d_1-d_3,d_2-d_3) \in G})
\]

\[= id_M \circ (W_{d \in D \rightarrow (d,d,d) \in D(3)} \circ W_{(d_1,d_2,d_3) \in D(3) \rightarrow (0,0,0,0,0,0,d_1-d_3,d_2-d_3) \in G})
\]

\[= id_M \circ (W_{d \in D \rightarrow (0,0,0,0,0,0,d_1,d_2,d_3) \in G})
\]

This completes the proof.

5 From the General Jacobi Identity to the Jacobi Identity

**Notation 35** We write

\[ (M^M \otimes W_D)_{id_M} \]
for the pullback of

\[(M^M \otimes W_D)^{id_M} \rightarrow M^M \otimes W_D \]

\[\downarrow \quad 1 \rightarrow \quad \downarrow M^M\]

where the right arrow \(M^M \otimes W_D \rightarrow M^M\) is the canonical projection, while the bottom arrow is the exponential transpose of \(id_M : 1 \times M = M \rightarrow M\).

**Theorem 36** The composition of morphisms

\[
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \xrightarrow{\text{Ass}^{1,1}_{M}} \left( M^M \otimes W_{D^2} \right)_{id_M}
\]

\[M^M \otimes W_{D^2} \rightarrow M^M \otimes W_{D(2)}\]

in succession is equivalent to the composition of morphisms

\[
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \xrightarrow{\text{Ass}^{1,1}_{M}} \left( M^M \otimes W_{D^2} \right)_{id_M}
\]

\[\text{id}_{M^M} \otimes W_{(d_1,d_2) \in D^2 \rightarrow (d_2,d_1) \in D^2} \left( M^M \otimes W_{D^2} \right)_{id_M}
\]

\[\left( M^M \otimes W_{D^2} \right)_{id_M} \rightarrow \left( M^M \otimes W_{D(2)} \right)_{id_M}\]

in succession, so that we have

\[
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \xrightarrow{\text{L}_M} \left( M^M \otimes W_{D^2} \right)_{id_M} \xrightarrow{\text{L}_M} \left( M^M \otimes W_D \right)_{id_M}
\]

which is equivalent to

\[
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \rightarrow \left( M^M \otimes W_D \right)_{id_M}\]

**Proof.** The nontrivial part of the statement is only the equivalence of (31) and (32), for which it is easy to modify the proof of Proposition 8 in §3.4 of [1].

The following proposition should be obvious.

**Proposition 37** We have the following two statements.
1. The composition of morphisms

\[
\left( M^1 \otimes W^2 \right)_{id_M} \times_{M^M \otimes W^D(2)} \left( M^2 \otimes W^2 \right)_{id_M} \times (M^M \otimes W^D)_{id_M}
\]

\[
\rightarrow \left( M^2 \otimes W^D \right)_{id_M} \times (M^M \otimes W^D)_{id_M} \xrightarrow{\text{Ass}^2_{M}} (M^M \otimes W^D^3)_{id_M}
\]

\[
(M^M \otimes W^D^3)_{id_M} \rightarrow (M^M \otimes W^D^3\{1,2\})_{id_M}
\]

in succession is equivalent to the composition of morphisms

\[
\left( M^1 \otimes W^2 \right)_{id_M} \times_{M^M \otimes W^D(2)} \left( M^2 \otimes W^2 \right)_{id_M} \times (M^M \otimes W^D)_{id_M}
\]

\[
\rightarrow \left( M^2 \otimes W^D \right)_{id_M} \times (M^M \otimes W^D)_{id_M} \xrightarrow{\text{Ass}^2_{M}} (M^M \otimes W^D^3)_{id_M}
\]

\[
(M^M \otimes W^D^3)_{id_M} \rightarrow (M^M \otimes W^D^3\{1,2\})_{id_M}
\]

in succession, so that we have the morphism

\[
\left( M^1 \otimes W^2 \right)_{id_M} \times_{M^M \otimes W^D(2)} \left( M^2 \otimes W^2 \right)_{id_M} \times (M^M \otimes W^D)_{id_M}
\]

\[
\xrightarrow{\text{Ass}^{\mathbf{1,2}}_{M}} (M^M \otimes W^D^3)_{id_M} \times_{M^M \otimes W^D^3\{1,2\}} (M^M \otimes W^D^3)_{id_M}
\]

\[
\xrightarrow{\zeta^3} (M^M \otimes W^D^3)_{id_M}
\]

which is equivalent to the morphism

\[
\left( M^1 \otimes W^2 \right)_{id_M} \times_{M^M \otimes W^D(2)} \left( M^2 \otimes W^2 \right)_{id_M} \times (M^M \otimes W^D)_{id_M}
\]

\[
\xrightarrow{\text{Ass}^{\mathbf{1,1}}_{M}} (M^M \otimes W^D^2)_{id_M}
\]

2. The composition of morphisms

\[
(M^M \otimes W^D)_{id_M} \times \left( M^1 \otimes W^2 \right)_{id_M} \times_{M^M \otimes W^D(2)} \left( M^2 \otimes W^2 \right)_{id_M}
\]

\[
\rightarrow (M^M \otimes W^D)_{id_M} \times (M^M \otimes W^D^2)_{id_M} \xrightarrow{\text{Ass}^{\mathbf{1,2}}_{M}} (M^M \otimes W^D^3)_{id_M}
\]

\[
(M^M \otimes W^D^3)_{id_M} \rightarrow (M^M \otimes W^D^3\{2,3\})_{id_M}
\]
in succession is equivalent to the composition of morphisms

\[
(M^M \otimes W_D)_{\text{id}_M} \times \left( (M^M \otimes W_{D^2})_{\text{id}_M} \times_{M^M \otimes W_{D^2}} (M^M \otimes W_{D^2})_{\text{id}_M} \right)
\]

\[
\rightarrow (M^M \otimes W_D)_{\text{id}_M} \times \left( (M^M \otimes W_{D^2})_{\text{id}_M} \right)
\]

\[
\text{Ass}_{M^1,2}^{1,2} (M^M \otimes W_{D^3})_{\text{id}_M}
\]

(36)

\[
(M^M \otimes W_{D^3})_{\text{id}_M} \rightarrow (M^M \otimes W_{D^3(2,3)})_{\text{id}_M}
\]

in succession, so that we have the morphism

\[
(M^M \otimes W_D)_{\text{id}_M} \times \left( (M^M \otimes W_{D^2})_{\text{id}_M} \times_{M^M \otimes W_{D^2}} (M^M \otimes W_{D^2})_{\text{id}_M} \right)
\]

\[
\rightarrow (M^M \otimes W_{D^2})_{\text{id}_M}
\]

which is equivalent to the morphism

\[
\zeta \cdot \left( (M^M \otimes W_{D^2})_{\text{id}_M} \right)
\]

\[
\text{Ass}_{M^1,1}^{1,1} (M^M \otimes W_{D^2})_{\text{id}_M}
\]

Notation 38 We introduce the following fifteen morphisms:

1.  
\[
\chi^{*1} : (M^M \otimes W_D)_{\text{id}_M} \times (M^M \otimes W_D)_{\text{id}_M} \times (M^M \otimes W_D)_{\text{id}_M} \rightarrow (M^M \otimes W_D)_{\text{id}_M}
\]

as the canonical projection.

2.  
\[
\chi^{*2} : (M^M \otimes W_D)_{\text{id}_M} \times (M^M \otimes W_D)_{\text{id}_M} \times (M^M \otimes W_D)_{\text{id}_M} \rightarrow (M^M \otimes W_D)_{\text{id}_M}
\]

as the canonical projection.
3. \( \chi^* : (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \rightarrow (M^M \otimes W_D)^{id_M} \)

as the canonical projection.

4. \( \chi^{*12} : (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \rightarrow (M^M \otimes W_D)^{id_M} \)

as

\[
\left( M^M \otimes W_D \right)^{id_M} \times \left( M^M \otimes W_D \right)^{id_M} \times \left( M^M \otimes W_D \right)^{id_M} \rightarrow \left( M^M \otimes W_D \right)^{id_M}
\]

5. \( \chi^{*21} : (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \rightarrow (M^M \otimes W_D)^{id_M} \)

as

\[
\left( M^M \otimes W_D \right)^{id_M} \times \left( M^M \otimes W_D \right)^{id_M} \times \left( M^M \otimes W_D \right)^{id_M} \rightarrow \left( M^M \otimes W_D \right)^{id_M}
\]

6. \( \chi^{*13} : (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \rightarrow (M^M \otimes W_D)^{id_M} \)

as

\[
\left( M^M \otimes W_D \right)^{id_M} \times \left( M^M \otimes W_D \right)^{id_M} \times \left( M^M \otimes W_D \right)^{id_M} \rightarrow \left( M^M \otimes W_D \right)^{id_M}
\]
7. \(\chi^{*31} : \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \)
\[\rightarrow (M^M \otimes W_{D^2})_{id_M}\]
as
\[\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \rightarrow \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \triangleleft \text{Ass}_{1,1} (M^M \otimes W_{D^2})_{id_M}\]
\[\text{id}_{M^M \otimes W_{(d_1,d_2)}} \rightarrow (M^M \otimes W_{D^2})_{id_M}\]

8. \(\chi^{*23} : \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \)
\[\rightarrow (M^M \otimes W_{D^2})_{id_M}\]
as
\[\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \rightarrow \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \triangleleft \text{Ass}_{1,1} (M^M \otimes W_{D^2})_{id_M}\]

9. \(\chi^{*32} : \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \)
\[\rightarrow (M^M \otimes W_{D^2})_{id_M}\]
as
\[\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \rightarrow \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \triangleleft \text{Ass}_{1,1} (M^M \otimes W_{D^2})_{id_M}\]
\[\text{id}_{M^M \otimes W_{(d_1,d_2)}} \rightarrow (M^M \otimes W_{D^2})_{id_M}\]

10. \(\chi^{*123} : \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \)
\[\rightarrow (M^M \otimes W_{D^3})_{id_M}\]
\[\begin{align*}
\text{as} \\
\text{Ass}^{1,1}_M : & \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_{D^3} \right)_{id_M}
\end{align*}\]

11.
\[\begin{align*}
\chi^{*32} : & \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_{D^3} \right)_{id_M}
\end{align*}\]

\[\begin{align*}
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_{D^3} \right)_{id_M}
\end{align*}\]

\[\begin{align*}
\text{as} \\
\text{Ass}^{1,1}_M : & \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_{D^3} \right)_{id_M}
\end{align*}\]

12.
\[\begin{align*}
\chi^{*231} : & \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_{D^3} \right)_{id_M}
\end{align*}\]

\[\begin{align*}
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_{D^3} \right)_{id_M}
\end{align*}\]

\[\begin{align*}
\text{as} \\
\text{Ass}^{1,1}_M : & \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_{D^3} \right)_{id_M}
\end{align*}\]

13.
\[\begin{align*}
\chi^{*31} : & \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_{D^3} \right)_{id_M}
\end{align*}\]

\[\begin{align*}
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
& \rightarrow \left( M^M \otimes W_{D^3} \right)_{id_M}
\end{align*}\]
14. \[ \chi^{*_{312}} : (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \rightarrow (M^M \otimes W_{D^3})_{id_M} \]

as

\[ (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \rightarrow (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \xrightarrow{ \text{Ass}^{1,1,1}_{M} } \]

\[ (M^M \otimes W_{D^3})_{id_M} \xrightarrow{id_M \otimes W_{(d_1,d_2,d_3) \in D^3 \rightarrow (d_3,d_1,d_2) \in D^3}} (M^M \otimes W_{D^3})_{id_M} \]

15. \[ \chi^{*_{231}} : (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \rightarrow (M^M \otimes W_{D^3})_{id_M} \]

as

\[ (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \rightarrow (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \xrightarrow{ \text{Ass}^{1,1,1}_{M} } \]

\[ (M^M \otimes W_{D^3})_{id_M} \xrightarrow{id_M \otimes W_{(d_1,d_2,d_3) \in D^3 \rightarrow (d_3,d_1,d_2) \in D^3}} (M^M \otimes W_{D^3})_{id_M} \]

Lemma 39 We have the following statements:

1. The composition of morphisms \( \chi^{*_{321}} \) and \( (M^M \otimes W_{D^3})_{id_M} \rightarrow (M^M \otimes W_{D^3}(2,3))_{id_M} \)

is equivalent to the composition of morphisms \( \chi^{*_{231}} \) and \( (M^M \otimes W_{D^3})_{id_M} \rightarrow (M^M \otimes W_{D^3}(2,3))_{id_M} \)

so that we have

\[ (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \]

\[ (\chi^{*_{321}}, \chi^{*_{231}}) (M^M \otimes W_{D^3}) \times_{M^M \otimes W_{D^3}(2,3)} (M^M \otimes W_{D^3})_{id_M} \xrightarrow{1} (M^M \otimes W_{D^3})_{id_M} \]

(37)
which is equivalent to
\[
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M}
\]
\[
\left( (\chi^{*2}, \chi^{*3}), \chi^{*1} \right)
\]
\[
\left( (M^M \otimes W_D)_{id_M} \times M^M \otimes W_{D\{2\}} \right) \times \left( M^M \otimes W_D \right)_{id_M}
\]
\[
\zeta^- \times id_{M^M \otimes W_D} \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \xrightarrow{Ass_{M^1}^{1,1}} \left( M^M \otimes W_D \right)_{id_M}
\]
\[
id_{M^M \otimes W_{(d_1,d_2)\in D^2\to (d_2,d_1)\in D^2}} \left( M^M \otimes W_D \right)_{id_M}
\]
\[
(38)
\]

2. The composition of morphisms \(\chi^{*123}\) and
\[
\left( M^M \otimes W_D \right)_{id_M} \rightarrow \left( M^M \otimes W_{D^3\{2,3\}} \right)_{id_M}
\]
is equivalent to the composition of morphisms \(\chi^{*123}\) and
\[
\left( M^M \otimes W_D \right)_{id_M} \rightarrow \left( M^M \otimes W_{D^3\{2,3\}} \right)_{id_M}
\]
so that we have
\[
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M}
\]
\[
\left( \chi^{*123}, \chi^{*123} \right) \left( M^M \otimes W_D^3 \right)_{id_M} \times M^M \otimes W_{D^3\{2,3\}} \left( M^M \otimes W_D^3 \right)_{id_M}
\]
\[
\zeta^- \left( M^M \otimes W_D^3 \right)_{id_M}
\]
\[
(39)
\]

which is equivalent to
\[
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M}
\]
\[
\left( \chi^{*1}, \left( \chi^{*2}, \chi^{*3} \right) \right) \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M}
\]
\[
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M}
\]
\[
id_{M^M \otimes W_D} \times \zeta^- \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \xrightarrow{Ass_{M^1}^{1,1}} \left( M^M \otimes W_D \right)_{id_M}
\]
\[
(40)
\]

3. The composition of (38) and
\[
\left( M^M \otimes W_D \right)_{id_M} \rightarrow \left( M^M \otimes W_D \right)_{id_M}
\]

45
is equivalent to the composition of (40) and
\[(M^M \otimes W_{D^3})_{id_M} \to (M^M \otimes W_{D(2)})_{id_M}\]
so that we have
\[
\left(\frac{M^M \otimes W_D}{1} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{2} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{3} \right)_{id_M}
\]
\[
\left(\frac{M^M \otimes W_{D^3}}{321} \right)_{id_M} \times \left(\frac{M^M \otimes W_{D^3(2,3)}}{132} \right)_{id_M} \times \left(\frac{M^M \otimes W_{D^3(2,3)}}{231} \right)_{id_M}
\]
\[
\zeta^2 \times \frac{M^M \otimes W_{D(2)}}{id_M} \times \left(\frac{M^M \otimes W_{D^3(2,3)}}{id_M} \right)
\]
which is equivalent to
\[
\left(\frac{M^M \otimes W_D}{1} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{2} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{3} \right)_{id_M} \times \left(\frac{id_{M^M \otimes W_D}}{id_M} \right) \times (L_M)
\]
\[
\left(\frac{M^M \otimes W_D}{1} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{2} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{3} \right)_{id_M} \times \left(\frac{id_{M^M \otimes W_D}}{id_M} \right) \times (L_M)
\]
\[
\left(\frac{M^M \otimes W_D}{1} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{2} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{3} \right)_{id_M} \times \left(\frac{id_{M^M \otimes W_D}}{id_M} \right) \times (L_M)
\]
\[
\left(\frac{M^M \otimes W_D}{1} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{2} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{3} \right)_{id_M} \times \left(\frac{id_{M^M \otimes W_D}}{id_M} \right) \times (L_M)
\]

**Proof.** The first and the second statements follow from Proposition [37]. The last statement follows from Theorem [36].

**Lemma 40** We have the following statements:

1. The composition of morphisms \(\chi^{*132}\) and
\[
(M^M \otimes W_{D^3})_{id_M} \to (M^M \otimes W_{D^3(1,3)})_{id_M}
\]
in succession is equivalent to the composition of morphisms \(\chi^{*312}\) and
\[
(M^M \otimes W_{D^3})_{id_M} \to (M^M \otimes W_{D^3(1,3)})_{id_M}
\]
in succession, so that we have
\[
\left(\frac{M^M \otimes W_D}{1} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{2} \right)_{id_M} \times \left(\frac{M^M \otimes W_D}{3} \right)_{id_M}
\]
\[
\left(\frac{\chi^{*132}, \chi^{*312}}{id_M} \right) \times \left(\frac{M^M \otimes W_{D^3(1,3)}}{id_M} \right)
\]
\[
\zeta^2 \left(\frac{M^M \otimes W_{D^3}}{id_M} \right)
\]
which is equivalent to

\[(M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \]

\[
\langle (\chi^{*13}, \chi^{*31}), \chi^{*2} \rangle \]

\[
(M^M \otimes W_{D^2})_{id_M} \times_{M^M \otimes W_{D^2}} (M^M \otimes W_{D^2})_{id_M} \times (M^M \otimes W_{D^2})_{id_M}
\]

\[
\zeta^- \times \text{id}_{M^M \otimes W_{D^2}} (M^M \otimes W_{D^2})_{id_M} \times (M^M \otimes W_{D^2})_{id_M} \quad \text{Ass}_{1,1} (M^M \otimes W_{D^2})_{id_M}
\]

\[
\text{id}_{M^M \otimes W_{(d_1, d_2) \in D^2 \rightarrow (d_2, d_1) \in D^2}} (M^M \otimes W_{D^2})_{id_M}
\]

\[
(44)
\]

\[\zeta^- \cdot \text{id}_{M^M \otimes W_{D^2}} (M^M \otimes W_{D^2})_{id_M}
\]

\[
(45)
\]

2. The composition of morphisms \(\chi^{*213}\) and

\[
(M^M \otimes W_{D^3})_{id_M} \rightarrow (M^M \otimes W_{D^3}(1,3))_{id_M}
\]

is equivalent to the composition of morphisms \(\chi^{*231}\) and

\[
(M^M \otimes W_{D^3})_{id_M} \rightarrow (M^M \otimes W_{D^3}(1,3))_{id_M}
\]

in succession, so that we have

\[
(M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M}
\]

\[
(\chi^{*213}, \chi^{*231}) (M^M \otimes W_{D^3})_{id_M} \times_{M^M \otimes W_{D^3}(1,3)} (M^M \otimes W_{D^3})_{id_M} \quad \zeta^- (M^M \otimes W_D)_{id_M}
\]

\[
(45)
\]

which is equivalent to

\[
(M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M}
\]

\[
(\chi^{*2}, (\chi^{*13}, \chi^{*31}))
\]

\[
(M^M \otimes W_D)_{id_M} \times ((M^M \otimes W_{D^2})_{id_M} \times_{M^M \otimes W_{D^2}} (M^M \otimes W_{D^2})_{id_M})
\]

\[
\text{id}_{M^M \otimes W_D} \times \zeta^- (M^M \otimes W_D)_{id_M} \times (M^M \otimes W_D)_{id_M} \quad \text{Ass}_{1,1} (M^M \otimes W_{D^2})_{id_M}
\]

\[
(46)
\]

3. The composition of \((44)\) and

\[
(M^M \otimes W_{D^2})_{id_M} \rightarrow (M^M \otimes W_{D^2(2)})_{id_M}
\]

is equivalent to the composition of \((46)\) and

\[
(M^M \otimes W_{D^2})_{id_M} \rightarrow (M^M \otimes W_{D^2(2)})_{id_M}
\]
so that we have

$$
\begin{align*}
&\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
&\left( (x^{*132}, x^{*312}), (x^{*213}, x^{*231}) \right)
\end{align*}
$$

which is equivalent to

$$
\begin{align*}
&\left( M^M \otimes W_{D^2} \right)_{id_M} \times \left( M^M \otimes W_{D^2} \right)_{id_M} \frac{\zeta}{\zeta^2} \frac{\zeta}{\zeta^2} \left( M^M \otimes W_{D^2} \right)_{id_M} \\
&\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \frac{\zeta}{\zeta^2} \frac{\zeta}{\zeta^2} \left( M^M \otimes W_D \right)_{id_M}
\end{align*}
$$

Proof. The first and the second statements follow from Proposition 37. The last statement follows from Theorem 36.

Lemma 41 We have the following statements:

1. The composition of morphisms $x^{*213}$ and

$$
\left( M^M \otimes W_{D^3} \right)_{id_M} \to \left( M^M \otimes W_{D^3(1,2)} \right)_{id_M}
$$

is equivalent to the composition of morphisms $x^{*123}$ and

$$
\left( M^M \otimes W_{D^3} \right)_{id_M} \to \left( M^M \otimes W_{D^3(1,2)} \right)_{id_M}
$$

so that we have

$$
\begin{align*}
&\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \\
&\left( x^{*213}, x^{*123} \right)
\end{align*}
$$

$$
\zeta \frac{\zeta}{\zeta^2} \left( M^M \otimes W_{D^2} \right)_{id_M}
$$

48
which is equivalent to

\[
\begin{align*}
(M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \\
((\chi^{*31}, \chi^{*12}), \chi^{*3}) \\
((M^M \otimes W_{D^2})^{id_M} \times (M^M \otimes W_{D^2})^{id_M} \times (M^M \otimes W_{D^2})^{id_M}) \\
(\chi^{*321}, \chi^{*312}) \\
(M^M \otimes W_{D^3})^{id_M} \times (M^M \otimes W_{D^3})^{id_M} \times (M^M \otimes W_{D^3})^{id_M} \\
\zeta^- \times (M^M \otimes W_{D})^{id_M} \times (M^M \otimes W_{D})^{id_M} \\
\text{id}_{M^M \otimes W_D} \times (M^M \otimes W_{D^2})^{id_M} \\
\zeta^3 \times (M^M \otimes W_{D^2})^{id_M} \\
\text{id}_{M^M \otimes W_D} \times (\text{id}_{M^M \otimes W_D} \times (M^M \otimes W_{D^2})^{id_M} \\
(50)
\end{align*}
\]

2. The composition of morphisms \(\chi^{*321}\) and

\[
(M^M \otimes W_{D^3})^{id_M} \rightarrow (M^M \otimes W_{D^3})^{(2,3)}
\]
is equivalent to the composition of morphisms \(\chi^{*312}\) and

\[
(M^M \otimes W_{D^3})^{id_M} \rightarrow (M^M \otimes W_{D^3})^{(2,3)}
\]

so that we have

\[
\begin{align*}
(M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \\
(\chi^{*321}, \chi^{*312}) \\
(M^M \otimes W_{D^3})^{id_M} \times (M^M \otimes W_{D^3})^{id_M} \times (M^M \otimes W_{D^3})^{id_M} \\
\zeta^- \times (M^M \otimes W_{D})^{id_M} \times (M^M \otimes W_{D})^{id_M} \\
\text{id}_{M^M \otimes W_D} \times (\text{id}_{M^M \otimes W_D} \times (M^M \otimes W_{D^2})^{id_M} \\
\zeta^3 \times (M^M \otimes W_{D^2})^{id_M} \\
\text{id}_{M^M \otimes W_D} \times (\text{id}_{M^M \otimes W_D} \times (M^M \otimes W_{D^2})^{id_M} \\
(51)
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
(M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \\
(\chi^{*21}, \chi^{*21}) \\
(M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \times (M^M \otimes W_D)^{id_M} \\
\text{id}_{M^M \otimes W_D} \times (M^M \otimes W_D)^{id_M} \\
\text{id}_{M^M \otimes W_D} \times (\text{id}_{M^M \otimes W_D} \times (M^M \otimes W_{D^2})^{id_M} \\
(52)
\end{align*}
\]

3. The composition of \(50\) and

\[
(M^M \otimes W_{D^2})^{id_M} \rightarrow (M^M \otimes W_{D^2})^{id_M}
\]
is equivalent to the composition of \(52\) and

\[
(M^M \otimes W_{D^2})^{id_M} \rightarrow (M^M \otimes W_{D^2})^{id_M}
\]
in succession, so that we have

\[
\begin{align*}
&\left( M^M \otimes W_1 \right)_{id_M} \times \left( M^M \otimes W_2 \right)_{id_M} \times \left( M^M \otimes W_3 \right)_{id_M} \\
&\left( (\chi^{*123}, \chi^{*123}), (\chi^{*121}, \chi^{*312}) \right)
\end{align*}
\]

\[
\frac{\zeta^3 \times_M \otimes_W \left( M^M \otimes W_2 \right)_{id_M} \times_M \otimes_W \left( M^M \otimes W_3 \right)_{id_M}}{(53)} \Rightarrow \left( M^M \otimes W_2 \right)_{id_M} \times_M \otimes_W \left( M^M \otimes W_3 \right)_{id_M} \left( M^M \otimes W_2 \right)_{id_M}
\]

which is equivalent to

\[
\begin{align*}
&\left( M^M \otimes W_1 \right)_{id_M} \times \left( M^M \otimes W_2 \right)_{id_M} \times \left( M^M \otimes W_3 \right)_{id_M} \\
&\left( M^M \otimes W_3 \right)_{id_M} \times \left( M^M \otimes W_1 \right)_{id_M} \times \left( M^M \otimes W_2 \right)_{id_M} \times_M \otimes_W \left( M^M \otimes W_3 \right)_{id_M} \times_L \left( M^M \otimes W_3 \right)_{id_M} \\
&\left( M^M \otimes W_3 \right)_{id_M} \times \left( M^M \otimes W_2 \right)_{id_M} \times \left( M^M \otimes W_3 \right)_{id_M} (54)
\end{align*}
\]

Proof. The first and the second statements follow from Proposition 37. The last statement follows from Theorem 36.

Theorem 42 (The conventional Jacobi Identity) We have the following two statements:

1. The three morphisms (41), (47) and (53) sum up only to vanish.
2. The three morphisms (42), (48) and (54) sum up only to vanish.
\textbf{Proof.} We apply the general Jacobi identity to the morphism

\[
\left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M} \times \left( M^M \otimes W_D \right)_{id_M}
\]

\[
\left( \begin{array}{c}
\chi^{231} \\
\times M \otimes W_D^3(1,2,3) \\
\chi^{132} \\
\times M \otimes W_D(2) \\
\chi^{213} \\
\times M \otimes W_D^3(1,2,3) \\
\chi^{321} \\
\times M \otimes W_D^3(2,3,1) \\
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\chi^{213} \\
\times M \otimes W_D^3(1,2,3) \\
\chi^{132} \\
\times M \otimes W_D(2) \\
\chi^{231} \\
\times M \otimes W_D^3(1,2,3) \\
\chi^{312} \\
\times M \otimes W_D^3(2,3,1) \\
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\chi^{321} \\
\times M \otimes W_D^3(2,3,1) \\
\chi^{132} \\
\times M \otimes W_D(2) \\
\chi^{312} \\
\times M \otimes W_D^3(2,3,1) \\
\chi^{213} \\
\times M \otimes W_D^3(1,2,3) \\
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\chi^{312} \\
\times M \otimes W_D^3(2,3,1) \\
\chi^{213} \\
\times M \otimes W_D(2) \\
\chi^{231} \\
\times M \otimes W_D^3(1,2,3) \\
\chi^{132} \\
\times M \otimes W_D(2) \\
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\chi^{132} \\
\times M \otimes W_D^3(2,3,1) \\
\chi^{213} \\
\times M \otimes W_D(2) \\
\chi^{231} \\
\times M \otimes W_D^3(1,2,3) \\
\chi^{321} \\
\times M \otimes W_D^3(2,3,1) \\
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\chi^{321} \\
\times M \otimes W_D^3(2,3,1) \\
\chi^{132} \\
\times M \otimes W_D(2) \\
\chi^{312} \\
\times M \otimes W_D^3(2,3,1) \\
\chi^{213} \\
\times M \otimes W_D^3(1,2,3) \\
\end{array} \right)
\]
so as to obtain the first statement. The second statement follows directly from the first by Lemmas 39, 40 and 41.

References


